

RELATIVE SINGULAR LOCUS AND MATRIX FACTORIZATIONS

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ABSTRACT. We introduce the notion of the relative singular locus $\text{Sing}(T/S)$ of a closed immersion $T \hookrightarrow S$ of Noetherian schemes, and for a separated Noetherian scheme X with ample family of line bundles and a non-zero-divisor $W \in \Gamma(X, L)$ of a line bundle L on X , we classify certain thick subcategories of the derived matrix factorization category $\text{DMF}(X, L, W)$ by means of specialization-closed subsets of relative singular locus $\text{Sing}(X_0/X)$ of the zero scheme $X_0 := W^{-1}(0) \subset X$. Furthermore, we show that the spectrum of the tensor triangulated category $(\text{DMF}(X, L, W), \otimes^{\frac{1}{2}})$ is homeomorphic to the relative singular locus $\text{Sing}(X_0/X)$ by using the classification result and the theory of Balmer's tensor triangular geometry.

1. INTRODUCTION

1.1. Background. For a given category of algebraic objects associated to a scheme, it is expected that we can extract geometric information of the scheme or the scheme itself from the category. Gabriel reconstructed a Noetherian scheme X from the abelian category $\text{Qcoh } X$ of quasi-coherent sheaves on X [Gab], and later Rosenberg generalized the reconstruction theorem for arbitrary schemes [Ros]. Although we can't reconstruct a smooth variety from the derived category of coherent sheaves in general, Balmer reconstructed arbitrary Noetherian scheme X from the tensor triangulated category $(\text{Perf } X, \otimes)$ of perfect complexes on X with the natural tensor structure \otimes [Bal]. Balmer's idea is to associate to any tensor triangulated category (\mathcal{T}, \otimes) a ringed space $\text{Spec}(\mathcal{T}, \otimes) = (\text{Spc}(\mathcal{T}, \otimes), \mathcal{O})$, and he proved an isomorphism $X \cong \text{Spec}(\text{Perf } X, \otimes)$ by using Thomason's result of classification of thick subcategories of perfect complexes $\text{Perf } X$ which are closed under \otimes -action of $\text{Perf } X$.

In addition to the Thomason's result, classifications of thick subcategories of triangulated categories are studied in many articles. For example, Takahashi classified thick subcategories of the stable category $\underline{\text{CM}}(R)$ of maximal Cohen-Macaulay modules over an abstract hypersurface local ring R [Tak]. Stevenson proved a classification of certain thick subcategories of the singularity category $\text{D}^{\text{sg}}(X)$ of a hypersurface singularity X [Ste].

1.2. Relative singular locus. To state our main results, we introduce a new notion of relative singular locus. Let $i : T \hookrightarrow S$ be a closed immersion of Noetherian schemes. We define the *relative singular locus*, denoted by $\text{Sing}(T/S)$, of i as the following subset of T ;

$$\text{Sing}(T/S) := \{p \in T \mid \exists F \in \text{coh } T \text{ such that } F_p \notin \text{Perf } \mathcal{O}_{T,p} \text{ and } i_*(F) \in \text{Perf } S\}.$$

We also consider the *locally relative singular locus* $\text{Sing}^{\text{loc}}(T/S)$ of i defined by

$$\text{Sing}^{\text{loc}}(T/S) := \{p \in T \mid \exists M \in \text{mod } \mathcal{O}_{T,p} \text{ such that } M \notin \text{Perf } \mathcal{O}_{T,p} \text{ and } i_{p*}(M) \in \text{Perf } \mathcal{O}_{S,p}\}.$$

By definition, we have the inclusions

$$\text{Sing}(T/S) \subseteq \text{Sing}^{\text{loc}}(T/S) \subseteq \text{Sing}(T),$$

where $\text{Sing}(T)$ is the usual singular locus of T . These loci can be different to each other, but, if S is regular, these loci are equal; $\text{Sing}(T/S) = \text{Sing}^{\text{loc}}(T/S) = \text{Sing}(T)$. Roughly speaking, the relative singular locus $\text{Sing}(T/S)$ is a set of points p in T such that the mildness of the singularity of p in T get worse than the mildness of the singularity of p in S . In fact, for a quasi-projective variety X over \mathbb{C} and a regular function $f \in \Gamma(X, \mathcal{O}_X)$ which is non-zero-divisor, we have the following equality of subsets of the associated complex analytic space $(X^{\text{an}}, \mathcal{O}_X^{\text{an}})$;

$$\text{Sing}^{\text{loc}}(f^{-1}(0)/X) \cap X^{\text{an}} = \text{Crit}(f^{\text{an}}) \cap \text{Zero}(f^{\text{an}}),$$

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where $\text{Crit}(f^{\text{an}})$ denotes the critical locus of the associated function $f^{\text{an}} \in \Gamma(X^{\text{an}}, \mathcal{O}_X^{\text{an}})$, which is defined by

$$\text{Crit}(f^{\text{an}}) := \{p \in X^{\text{an}} \mid (f^{\text{an}})_p - f^{\text{an}}(p) \in \mathfrak{m}_p^2\},$$

and $\text{Zero}(f^{\text{an}})$ is the zero locus of f^{an} .

1.3. Main results. A data (X, L, W) is called *Landau-Ginzburg model*, or just *LG-model*, if X is a scheme, L is a line bundle on X , and $W \in \Gamma(X, L)$ is a section of L . To a LG-model (X, L, W) we associate a triangulated category $\text{DMF}(X, L, W)$, called *derived matrix factorization category*, introduced by Positselski [Pos, EP]. Tensor products of matrix factorizations defines the bifunctor;

$$(-) \otimes (-) : \text{DMF}(X, L, W_1) \times \text{DMF}(X, L, W_2) \rightarrow \text{DMF}(X, L, W_1 + W_2).$$

In particular, $\text{DMF}(X, L, W)$ has a tensor action from $\text{DMF}(X, L, 0)$.

The following is our main result of classification of thick subcategories of derived matrix factorization categories.

Theorem 1.1 (Theorem 5.6). *Let X be a separated Noetherian scheme with ample family of line bundles, L be a line bundle on X , and let $W \in \Gamma(X, L)$ be a non-zero-divisor. Denote by X_0 the zero scheme of W . Then there is a bijective correspondence*

$$\left\{ \begin{array}{l} \text{unions of closed} \\ \text{subsets of } \text{Sing}(X_0/X) \end{array} \right\} \xrightarrow{\sigma} \left\{ \begin{array}{l} \text{thick subcategories of } \text{DMF}(X, L, W) \text{ that are} \\ \text{closed under tensor action from } \text{DMF}(X, L, 0) \end{array} \right\}$$

The bijective map σ sends Y to the thick subcategory consisting of matrix factorizations $F \in \text{DMF}(X, L, W)$ with $\text{Supp}(F) \subseteq Y$. The inverse bijection τ sends \mathcal{T} to the union $\bigcup_{F \in \mathcal{T}} \text{Supp}(F)$.

If X is a regular separated Noetherian scheme, then X has an ample family of line bundles and $\text{DMF}(X, L, W)$ is equivalent to the singularity category $\text{D}^{\text{sg}}(X_0)$. Furthermore, if $X = \text{Spec } R$ is affine with R regular local, $\text{DMF}(X, L, W)$ is equivalent to the stable category $\underline{\text{CM}}(R/W)$ of maximal Cohen-Macaulay modules over the hypersurface R/W . Hence Theorem 1.1 can be considered as a simultaneous generalization of Stevenson's result in [Ste] and Takahashi's result in [Tak].

As an application of the above main result, we see that the closedness of the relative singular locus $\text{Sing}(X_0/X)$ is related to the existence of a \otimes -generator of $\text{DMF}(X, L, W)$, where we say that an object $G \in \text{DMF}(X, L, W)$ is a \otimes -generator if the smallest thick subcategory that is closed under tensor action from $\text{DMF}(X, L, 0)$ and contains G is $\text{DMF}(X, L, W)$.

Corollary 1.2 (Corollary 5.8). *Notation is same as in Theorem 1.1. Then the subset $\text{Sing}(X_0/X)$ of X_0 is closed if and only if $\text{DMF}(X, L, W)$ has a \otimes -generator.*

Furthermore, we construct the relative singular loci from the derived matrix factorization categories. If $2 \in \Gamma(X, \mathcal{O}_X)$ is a unit in the ring $\Gamma(X, \mathcal{O}_X)$, the derived matrix factorization category $\text{DMF}(X, L, W)$ has a natural (pseudo) tensor triangulated structure $\otimes^{\frac{1}{2}}$ on it. Using Theorem 1.1 and the theory of Balmer's tensor triangular geometry, we prove that the spectrum of the (pseudo) tensor triangulated category $(\text{DMF}(X, L, W), \otimes^{\frac{1}{2}})$ is the relative singular locus $\text{Sing}(X_0/X)$.

Theorem 1.3 (Corollary 6.10). *Let X be a separated Noetherian scheme with an ample family of line bundles, and let $W \in \Gamma(X, L)$ be a non-zero-divisor of a line bundle L . Assume that $2 \in \Gamma(X, \mathcal{O}_X)$ is a unit. Then we have a homeomorphism*

$$\text{Spc}(\text{DMF}(X, L, W), \otimes^{\frac{1}{2}}) \cong \text{Sing}(X_0/X).$$

This result is a generalization of Yu's result [Yu2, Theorem 1.2], where he proves Theorem 1.3 in the case when X is an affine regular scheme of finite Krull dimension by using the classification result due to Walker.

1.4. Plan of the paper. In section 2 we provide basic definitions and properties about derived matrix factorization categories. In section 3 we give the definitions of globally/locally relative singular loci and prove some properties about relative singular loci for zero schemes of regular sections of line bundles. In section 4 we prove tensor nilpotence properties of matrix factorizations which is key properties for our classification result. In section 5 we prove the main result Theorem 1.1. In section 6 we recall the theory of Balmer's tensor triangular geometry, and we study the natural tensor triangulated structure on derived matrix factorization categories.

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2. DERIVED MATRIX FACTORIZATIONS

2.1. Derived matrix factorization categories. In the first subsection, we recall the definition of the derived matrix factorization category of a Landau-Ginzburg model, which is introduced by Positselski (cf. [Pos], [EP]), and provide its basic properties.

Definition 2.1. A **Landau-Ginzburg model**, or **LG model**, is data (X, L, W) consisting of a scheme X , an invertible sheaf L on X , and a section $W \in \Gamma(X, L)$ of L .

Notation 2.2. If L is isomorphic to the structure sheaf \mathcal{O}_X , we denote the LG model by (X, W) . If $X = \text{Spec } R$ is an affine scheme, we denote the LG model by (R, L, W) , where L is considered as an invertible R -module and $W \in L$.

For a LG model, we consider its factorizations which are “twisted” complexes.

Definition 2.3. Let (X, L, W) be a LG model. A **factorization** F of (X, L, W) is a sequence

$$F = \left(F_1 \xrightarrow{\varphi_1^F} F_0 \xrightarrow{\varphi_0^F} F_1 \otimes L \right),$$

where each F_i is a coherent sheaf on X and each φ_i^F is a homomorphism such that $\varphi_0^F \circ \varphi_1^F = W \cdot \text{id}_{F_1}$ and $(\varphi_1^F \otimes L) \circ \varphi_0^F = W \cdot \text{id}_{F_0}$. Coherent sheaves F_0 and F_1 in the above sequence are called **components** of the factorization F . If the components F_i of F are locally free sheaves, we call F a **matrix factorization** of (X, L, W) .

Notation 2.4. We can consider any coherent sheaf $F \in \text{coh } X$ as a factorization of $(X, L, 0)$ of the following form

$$(0 \longrightarrow F \longrightarrow 0).$$

By abuse of notation, we will often denote the above factorization by the same notation F .

Definition 2.5. For a LG model (X, L, W) , we define an exact category

$$\text{coh}(X, L, W)$$

whose objects are factorizations of (X, L, W) , and whose set of morphisms are defined as follows: For two objects $E, F \in \text{coh}(X, L, W)$, we define $\text{Hom}(E, F)$ as the set of pairs (f_1, f_0) of $f_i \in \text{Hom}_{\text{coh } X}(E_i, F_i)$ such that the following diagram is commutative;

$$\begin{array}{ccccc} E_1 & \xrightarrow{\varphi_1^E} & E_0 & \xrightarrow{\varphi_0^E} & E_1 \otimes L \\ f_1 \downarrow & & \downarrow f_0 & & \downarrow f_1 \otimes L \\ F_1 & \xrightarrow{\varphi_1^F} & F_0 & \xrightarrow{\varphi_0^F} & F_1 \otimes L \end{array}$$

Note that if X is Noetherian, $\text{coh}(X, L, W)$ is an abelian category. We define a full additive subcategory

$$\text{MF}(X, L, W)$$

of $\text{coh}(X, L, W)$ whose objects are matrix factorizations. By construction, $\text{MF}(X, L, W)$ is also an exact category.

Since factorizations are “twisted” complexes, we can consider homotopy category of factorizations.

Definition 2.6. Two morphisms $f = (f_1, f_0)$ and $g = (g_1, g_0)$ in $\text{Hom}_{\text{coh}(X, L, W)}(E, F)$ are **homotopy equivalent**, denoted by $f \sim g$, if there exist two homomorphisms in $\text{coh } X$

$$h_0 : E_0 \rightarrow F_1 \quad \text{and} \quad h_1 : E_1 \otimes L \rightarrow F_0$$

such that $f_0 - g_0 = \varphi_1^F h_0 + h_1 \varphi_0^E$ and $f_1 \otimes L - g_1 \otimes L = \varphi_0^F h_1 + (h_0 \otimes L)(\varphi_1^E \otimes L)$.

The **homotopy category of factorizations**

$$\mathrm{Kcoh}(X, L, W)$$

is defined as the category whose objects are same as $\mathrm{coh}(X, L, W)$, and the set of morphisms are defined as the set of homotopy equivalence classes;

$$\mathrm{Hom}_{\mathrm{Kcoh}(X, L, W)}(E, F) := \mathrm{Hom}_{\mathrm{coh}(X, L, W)}(E, F) / \sim.$$

Similarly, we define the homotopy category of matrix factorizations $\mathrm{KMF}(X, L, W)$, i.e.

$$\mathrm{Ob}(\mathrm{KMF}(X, L, W)) := \mathrm{MF}(X, L, W)$$

$$\mathrm{Hom}_{\mathrm{KMF}(X, L, W)}(E, F) := \mathrm{Hom}_{\mathrm{MF}(X, L, W)}(E, F) / \sim.$$

Next we define the totalization of a bounded complex of factorizations, which is an analogy of the total complex of a double complex.

Definition 2.7. Let $F^\bullet = (\cdots \rightarrow F^i \xrightarrow{\delta^i} F^{i+1} \rightarrow \cdots)$ be a bounded complex of $\mathrm{coh}(X, L, W)$. For $l = 0, 1$, set

$$T_l := \bigoplus_{i+j=-l} F_j^i \otimes L^{\otimes [j/2]},$$

and let

$$t_l : T_l \rightarrow T_{l+1}$$

be a homomorphism given by

$$t_l|_{F_j^i \otimes L^{\otimes [j/2]}} := \delta_j^i \otimes L^{\otimes [j/2]} + (-1)^i \varphi_j^{F^i} \otimes L^{\otimes [j/2]},$$

where \bar{n} is n modulo 2, and $[m]$ is the minimum integer which is greater than or equal to a real number m . We define the **totalization** $\mathrm{Tot}(F^\bullet) \in \mathrm{coh}(X, L, W)$ of F^\bullet as

$$\mathrm{Tot}(F^\bullet) := (T_1 \xrightarrow{t_1} T_0 \xrightarrow{t_0} T_1 \otimes L).$$

In what follows, we will recall that the homotopy categories $\mathrm{Kcoh}(X, L, W)$ and $\mathrm{KMF}(X, L, W)$ have structures of triangulated categories.

Definition 2.8. We define an automorphism T on $\mathrm{Kcoh}(X, L, W)$, which is called **shift functor**, as follows. For an object $F \in \mathrm{Kcoh}(X, L, W)$, we define an object $T(F)$ as

$$T(F) := (F_0 \xrightarrow{-\varphi_0^F} F_1 \otimes L \xrightarrow{-\varphi_1^F \otimes L} F_0 \otimes L),$$

and for a morphism $f = (f_1, f_0) \in \mathrm{Hom}(E, F)$ we set $T(f) := (f_0, f_1 \otimes L) \in \mathrm{Hom}(T(E), T(F))$. For any integer $n \in \mathbb{Z}$, denote by $(-)[n]$ the functor $T^n(-)$.

Definition 2.9. Let $f : E \rightarrow F$ be a morphism in $\mathrm{coh}(X, L, W)$. We define its **mapping cone** $\mathrm{Cone}(f)$ to be the totalization of the complex

$$(\cdots \rightarrow 0 \rightarrow E \xrightarrow{f} F \rightarrow 0 \rightarrow \cdots)$$

with F in degree zero.

A **distinguished triangle** is a sequence in $\mathrm{Kcoh}(X, L, W)$ which is isomorphic to a sequence of the form

$$E \xrightarrow{f} F \xrightarrow{i} \mathrm{Cone}(f) \xrightarrow{p} E[1],$$

where i and p are natural injection and projection respectively.

The following proposition is well known to experts.

Proposition 2.10. *The homotopy categories $\mathrm{Kcoh}(X, L, W)$ and $\mathrm{KMF}(X, L, W)$ are triangulated categories with respect to the above shift functor and the above distinguished triangles.*

Following Positselski ([Pos], [EP]), we define derived factorization categories.

Definition 2.11. Denote by $\text{Acoh}(X, L, W)$ the smallest thick subcategory of $\text{Kcoh}(X, L, W)$ containing all totalizations of short exact sequences in $\text{coh}(X, L, W)$. We define the **derived factorization category** of (X, L, W) as the Verdier quotient

$$\text{Dcoh}(X, L, W) := \text{Kcoh}(X, L, W) / \text{Acoh}(X, L, W).$$

Similarly, we consider the thick subcategory $\text{AMF}(X, L, W)$ containing all totalizations of short exact sequences in the exact category $\text{MF}(X, L, W)$, and define the **derived matrix factorization category** by

$$\text{DMF}(X, L, W) := \text{KMF}(X, L, W) / \text{AMF}(X, L, W).$$

The following proposition is a special case of [BDFIK, Lemma 2.24].

Proposition 2.12 (cf. [BDFIK, Lemma 2.24]). *Assume that $X = \text{Spec } R$ is an affine scheme. For $P \in \text{KMF}(R, L, W)$ and $A \in \text{Acoh}(R, L, W)$, we have*

$$\text{Hom}_{\text{Kcoh}(R, L, W)}(P, A) = 0.$$

In particular, the Verdier localizing functor

$$\text{KMF}(R, L, W) \xrightarrow{\sim} \text{DMF}(R, L, W)$$

is an equivalence.

For later use, we consider larger categories of factorizations. Denote by $\text{Sh}(X, L, W)$ the abelian category whose objects are factorizations whose components are \mathcal{O}_X -modules. More precisely, objects of $\text{Sh}(X, L, W)$ are sequences of the following form

$$F = \left(F_1 \xrightarrow{\varphi_1^F} F_0 \xrightarrow{\varphi_0^F} F_1 \otimes L \right),$$

where F_i are \mathcal{O}_X -modules and φ_i^F are homomorphisms such that $\varphi_0^F \circ \varphi_1^F = W \cdot \text{id}_{F_1}$ and $\varphi_1^F \otimes L \circ \varphi_0^F = W \cdot \text{id}_{F_0}$. Denote by $\text{Qcoh}(X, L, W)$, $\text{InjSh}(X, L, W)$, and $\text{InjQcoh}(X, L, W)$ the full subcategories of $\text{Sh}(X, L, W)$ consisting of factorizations whose components are quasi-coherent sheaves, injective \mathcal{O}_X -modules, and injective quasi-coherent sheaves respectively.

Then, similarly to $\text{Kcoh}(X, L, W)$, we can consider their homotopy categories $\text{KSh}(X, L, W)$, $\text{KQcoh}(X, L, W)$, $\text{KInjSh}(X, L, W)$, $\text{KInjQcoh}(X, L, W)$ respectively, and these homotopy categories have natural triangulated structures similar to $\text{Kcoh}(X, L, W)$.

Definition 2.13. Denote by $\text{A}^\text{co}\text{Sh}(X, L, W)$ (resp. $\text{A}^\text{co}\text{Qcoh}(X, L, W)$) the smallest thick subcategory of $\text{KSh}(X, L, W)$ (resp. $\text{KQcoh}(X, L, W)$) containing all totalizations of short exact sequences in $\text{Sh}(X, L, W)$ (resp. $\text{Qcoh}(X, L, W)$) and closed under arbitrary direct sums. Following [Pos], [EP], we define the **coderived factorization categories** $\text{D}^\text{co}\text{Sh}(X, L, W)$ and $\text{D}^\text{co}\text{Qcoh}(X, L, W)$ as the following Verdier quotients

$$\text{D}^\text{co}\text{Sh}(X, L, W) := \text{KSh}(X, L, W) / \text{A}^\text{co}\text{Sh}(X, L, W)$$

$$\text{D}^\text{co}\text{Qcoh}(X, L, W) := \text{KQcoh}(X, L, W) / \text{A}^\text{co}\text{Qcoh}(X, L, W).$$

Lemma 2.14 ([BDFIK], [EP]). *Assume that X is Noetherian.*

- (1) *The natural functor $\text{KInjSh}(X, L, W) \rightarrow \text{D}^\text{co}\text{Sh}(X, L, W)$ is an equivalence.*
- (2) *The natural functor $\text{KInjQcoh}(X, L, W) \rightarrow \text{D}^\text{co}\text{Qcoh}(X, L, W)$ is an equivalence.*
- (3) *The natural functor $\text{D}^\text{co}\text{Qcoh}(X, L, W) \rightarrow \text{D}^\text{co}\text{Sh}(X, L, W)$ is fully faithful.*
- (4) *The natural functor $\text{Dcoh}(X, L, W) \rightarrow \text{D}^\text{co}\text{Qcoh}(X, L, W)$ is fully faithful.*
- (5) *The natural functor $\text{DMF}(X, L, W) \rightarrow \text{Dcoh}(X, L, W)$ is fully faithful.*

Proof. (1) and (2) follow from [BDFIK, Corollary 2.25]. (3) follows from (1) and (2). (4) and (5) are [EP, Propostion 1.5.(d)] and [EP, Corollary 2.3.(i)] respectively. \square

2.2. Case when $W = 0$. In this section, we consider cases when $W = 0$. Firstly, we will define cohomologies of factorizations of $(X, L, 0)$.

Definition 2.15. For an object $F \in \text{Qcoh}(X, L, 0)$, we define its cohomologies $H_i(F) \in \text{Qcoh } X$ as

$$H_i(F) := \text{Ker}(\varphi_i^F) / \text{Im}(\varphi_{i-1}^F) \quad \text{for } i \in \mathbb{Z}/2\mathbb{Z}$$

Lemma 2.16. *Let k be any field. Then, for any object $F \in \text{KMF}(k, 0)$, there are two finite dimensional k -vector spaces V_1 and V_2 such that F is isomorphic to $V_1 \oplus V_2[1]$ in $\text{KMF}(k, 0)$, where V_i denotes the factorization of the form $(0 \rightarrow V_i \rightarrow 0)$ by Notation 2.4.*

Proof. By [BDFIK, Lemma 2.26], there are two finite dimensional k -vector spaces V and V' , and a triangle of the following form in $\text{Dcoh}(k, 0) = \text{DMF}(k, 0)$

$$V \rightarrow V' \rightarrow F \rightarrow V[1].$$

But $\text{DMF}(k, 0) = \text{KMF}(k, 0)$ by Proposition 2.12, so we have a k -linear homomorphism $f : V \rightarrow V'$ such that F is isomorphic to $C(\bar{f})$, where \bar{f} is the morphism in $\text{KMF}(k, 0)$ represented by the following morphism in $\text{MF}(k, 0)$

$$\begin{array}{ccccc} 0 & \longrightarrow & V & \longrightarrow & 0 \\ \downarrow & & \downarrow f & & \downarrow \\ 0 & \longrightarrow & V' & \longrightarrow & 0 \end{array}$$

By construction of mapping cones, $C(\bar{f})$ is isomorphic to the following matrix factorization

$$\left(V \xrightarrow{f} V' \xrightarrow{0} V \right).$$

Let $I := \text{Im}(f)$ be the image of f , and let $K := \text{Ker}(f)$ be the kernel of f . Then there is a k -vector space J such that $V' = I \oplus J$. Since $V = K \oplus I$, we have the following isomorphism in $\text{MF}(k, 0)$

$$F \cong \left(K \xrightarrow{0} 0 \xrightarrow{0} K \right) \oplus \left(I \xrightarrow{\sim} I \xrightarrow{0} I \right) \oplus \left(0 \xrightarrow{0} J \xrightarrow{0} 0 \right)$$

But the object $\left(I \xrightarrow{\sim} I \xrightarrow{0} I \right)$ is zero in $\text{KMF}(k, 0)$. Hence $F \cong J \oplus K[1]$ in $\text{KMF}(k, 0)$. \square

Corollary 2.17. *Let k be a field. Any non-zero morphism $f : (0 \rightarrow k \rightarrow 0) \rightarrow E$ in $\text{KMF}(k, 0)$ is a split mono.*

Proof. This follows from Lemma 2.16. \square

2.3. Tensor products and sheaf Homs functors. In this subsection, we recall tensor products and local homs on derived matrix factorization categories. Let (X, L, W) be a LG model, and $V \in \Gamma(X, L)$ be another global section.

For $E \in \text{MF}(X, L, V)$ and $F \in \text{MF}(X, L, W)$, we define the tensor product

$$E \otimes F \in \text{MF}(X, L, V + W)$$

of E and F as

$$\begin{aligned} (E \otimes F)_1 &:= (E_1 \otimes F_0) \oplus (E_0 \otimes F_1), \\ (E \otimes F)_0 &:= (E_0 \otimes F_0) \oplus (E_1 \otimes F_1 \otimes L), \end{aligned}$$

$$\varphi_1^{E \otimes F} := \begin{pmatrix} \varphi_1^E \otimes 1 & 1 \otimes \varphi_1^F \\ -1 \otimes \varphi_0^F & \varphi_0^E \otimes 1 \end{pmatrix},$$

and

$$\varphi_0^{E \otimes F} := \begin{pmatrix} \varphi_0^E \otimes 1 & -1 \otimes \varphi_1^F \\ 1 \otimes \varphi_0^F & \varphi_1^E \otimes 1 \end{pmatrix}.$$

This defines an additive functor $(-) \otimes (-) : \text{MF}(X, L, V) \times \text{MF}(X, L, W) \rightarrow \text{MF}(X, L, V + W)$, and it naturally induces an exact functor

$$(-) \otimes (-) : \text{DMF}(X, L, V) \times \text{DMF}(X, L, W) \rightarrow \text{DMF}(X, L, V + W).$$

We define the sheaf $\mathcal{H}om$

$$\mathcal{H}om(E, F) \in \mathbf{MF}(X, L, W - V)$$

from E to F as

$$\mathcal{H}om(E, F)_1 := (\mathcal{H}om(E_1, F_0) \otimes L^{-1}) \oplus \mathcal{H}om(E_0, F_1),$$

$$\mathcal{H}om(E, F)_0 := \mathcal{H}om(E_0, F_0) \oplus \mathcal{H}om(E_1, F_1),$$

$$\varphi_1^{\mathcal{H}om(E, F)} := \begin{pmatrix} (*) \circ \varphi_0^E & \varphi_1^F \circ (*) \\ (\varphi_0^F \otimes L^{-1}) \circ (*) & (*) \circ \varphi_1^E \end{pmatrix},$$

and

$$\varphi_0^{\mathcal{H}om(E, F)} := \begin{pmatrix} -(*) \circ \varphi_1^E & \varphi_1^F \circ (*) \\ \varphi_0^F \circ (*) & -(*) \circ (\varphi_0^E \otimes L^{-1}) \end{pmatrix}.$$

This defines an additive functor $\mathcal{H}om(-, -) : \mathbf{MF}(X, L, V)^{\text{op}} \times \mathbf{MF}(X, L, W) \rightarrow \mathbf{MF}(X, L, W - V)$, and it induces an exact functor

$$\mathcal{H}om(-, -) : \mathbf{DMF}(X, L, V)^{\text{op}} \times \mathbf{DMF}(X, L, W) \rightarrow \mathbf{DMF}(X, L, W - V).$$

The following is standard, so we skip the proof (see [BFK] or [LS] for details).

Proposition 2.18. *Let $E \in \mathbf{MF}(X, L, V)$, $F \in \mathbf{MF}(X, L, W)$, and $G \in \mathbf{MF}(X, L, V + W)$.*

(1) *We have a natural isomorphism*

$$\text{Hom}_{\mathbf{DMF}(X, L, V+W)}(E \otimes F, G) \cong \text{Hom}_{\mathbf{DMF}(X, L, V)}(E, \mathcal{H}om(F, G)).$$

(2) *There is a natural isomorphism in $\mathbf{MF}(X, L, V + W)$*

$$\mathcal{H}om(G, E \otimes F) \cong \mathcal{H}om(G, E) \otimes F.$$

Recall that $\mathcal{O}_X \in \mathbf{MF}(X, L, 0)$ denotes the matrix factorization of the form $(0 \rightarrow \mathcal{O}_X \rightarrow 0)$ by Notation 2.4. For any object $F \in \mathbf{MF}(X, L, W)$, we define **the dual**

$$F^\vee := \mathcal{H}om(F, \mathcal{O}_X) \in \mathbf{MF}(X, L, -W)$$

of F . By Proposition 2.18, the functors $(-) \otimes F : \mathbf{DMF}(X, L, V) \rightarrow \mathbf{DMF}(X, L, V + W)$ and $(-) \otimes F^\vee : \mathbf{DMF}(X, L, V + W) \rightarrow \mathbf{DMF}(X, L, V)$ are adjoint;

$$(-) \otimes F \dashv (-) \otimes F^\vee.$$

2.4. Supports of matrix factorizations. We study the supports of objects in derived matrix factorization categories. Let (X, L, W) be a LG model.

For any point $p \in X$, we denote by $X_p := \text{Spec}(\mathcal{O}_{X,p})$ the stalk of X at p , and we consider the functor of taking stalks at p ,

$$(-)_p : \mathbf{DMF}(X, L, W) \rightarrow \mathbf{KMF}(X_p, W_p),$$

given by

$$F_p := \left((F_1)_p \xrightarrow{(\varphi_1)_p} (F_0)_p \xrightarrow{(\varphi_0)_p} (F_1)_p \right).$$

Definition 2.19. For an object $F \in \mathbf{DMF}(X, L, W)$, we define its support as

$$\text{Supp}(F) := \{p \in X \mid F_p \neq 0 \in \mathbf{KMF}(X_p, W_p)\}.$$

Proposition 2.20. *Let $F \in \mathbf{DMF}(X, L, W)$ be an object.*

(1) *If $X = \bigcup_{i \in I} U_i$ is an open covering of X , we have the equality of subsets of X*

$$\text{Supp}(F) = \bigcup_{i \in I} \text{Supp}(F|_{U_i}),$$

where $F|_{U_i}$ is the restriction of F to $\mathbf{DMF}(U_i, L|_{U_i}, W|_{U_i})$.

(2) *$\text{Supp}(F)$ is a closed subset of X .*

Proof. (1) This follows from isomorphisms $F_p \cong (F|_{U_i})_p$ for any $p \in U_i$.
 (2) We show the following equality

$$\mathrm{Supp}(F)^c = \bigcup_{U \in \mathcal{U}} U,$$

where $\mathcal{U} := \{ U \mid U \text{ is an open subscheme of } X \text{ such that } F|_U = 0 \text{ in } \mathrm{DMF}(U, L|_U, W|_U) \}$. The inclusion $\mathrm{Supp}(F)^c \supset \bigcup_{U \in \mathcal{U}} U$ is obvious. We verify that $\mathrm{Supp}(F)^c \subseteq \bigcup_{U \in \mathcal{U}} U$. By definition, for any $p \in \mathrm{Supp}(F)^c$, $F_p = 0$ in $\mathrm{KMF}(X_p, W_p)$. Let $h = (h_1, h_0)$ be a homotopy giving the homotopy equivalence $\mathrm{id}_{F_p} \sim 0$. Then there is a neighborhood U of p such that there exist morphisms $\overline{h}_1 : F_1|_U \otimes L_U \rightarrow F_0|_U$ and $\overline{h}_0 : F_0|_U \rightarrow F_1|_U$ in $\mathrm{coh} U$ with $\overline{h}_1|_p = h_1$ and $\overline{h}_0|_p = h_0$. Furthermore, since $\mathrm{id}_{(F_0)_p} - (\varphi_1)_p h_0 - h_1 (\varphi_0)_p = 0$ in $\mathrm{coh} U_p$, there exists an open neighborhood $V \subseteq U$ of p such that $\mathrm{id}_{F_0|_V} - \varphi_1|_V \overline{h}_1|_V - \varphi_0|_V \overline{h}_0|_V = 0$. Then $h_V = (\overline{h}_1|_V, \overline{h}_0|_V)$ gives a homotopy equivalence $\mathrm{id}_{F|_V} \sim 0$. Hence $F|_V = 0$ in $\mathrm{KMF}(V, L|_V, W|_V)$, in particular, so is in $\mathrm{DMF}(V, L|_V, W|_V)$. Therefore, we have $V \in \mathcal{U}$, which implies that $p \in \bigcup_{U \in \mathcal{U}} U$. \square

Definition 2.21. Let $F \in \mathrm{DMF}(X, L, W)$ be an object. For any point $p \in X$, let $\iota_p : \mathrm{Spec} k(p) \rightarrow X$ be a natural morphism, where $k(p) := \mathcal{O}_{X,p}/\mathfrak{m}_p$ is the residue field of the local ring $(\mathcal{O}_{X,p}, \mathfrak{m}_p)$. Then we denote by $W \otimes k(p)$ the pull-back $\iota_p^* W$, and we set

$$F \otimes k(p) := \iota_p^* F = \left(\iota_p^* F_1 \xrightarrow{\iota_p^* \varphi_1} \iota_p^* F_0 \xrightarrow{\iota_p^* \varphi_0} \iota_p^* F_1 \right) \in \mathrm{KMF}(k(p), W \otimes k(p)).$$

Then $(-) \otimes k(p)$ defines an exact functor

$$(-) \otimes k(p) : \mathrm{DMF}(X, L, W) \rightarrow \mathrm{KMF}(k(p), W \otimes k(p)).$$

The following lemma is a version of Nakayama's lemma for matrix factorizations.

Lemma 2.22. Let $F \in \mathrm{DMF}(X, L, W)$ be an object, and let $p \in X$ be a point. Then

$$p \in \mathrm{Supp}(F) \quad \text{if and only if} \quad F \otimes k(p) \neq 0 \text{ in } \mathrm{KMF}(k(p), W \otimes k(p)).$$

Proof. Note that the following diagram of functors is commutative

$$\begin{array}{ccc} \mathrm{DMF}(X, L, W) & \xrightarrow{(-) \otimes k(p)} & \mathrm{KMF}(k(p), W \otimes k(p)) \\ & \searrow (-)_p & \nearrow (-) \otimes k(\mathfrak{m}_p) \\ & \mathrm{KMF}(X_p, W_p) & \end{array}$$

where $\mathfrak{m}_p \in X_p$ is the unique closed point. Hence, if $F_p = 0$ in $\mathrm{KMF}(X_p, W_p)$, then $F \otimes k(p) = 0$ in $\mathrm{KMF}(k(p), W \otimes k(p))$.

For the other implication, it suffices to show that for a local ring (R, \mathfrak{m}) , an element $w \in R$, and an object $E \in \mathrm{KMF}(R, w)$, if $E \otimes_R R/\mathfrak{m} = 0$ in $\mathrm{KMF}(R/\mathfrak{m}, w \otimes R/\mathfrak{m})$, then $E = 0$ in $\mathrm{KMF}(R, w)$. Since R is local, any locally free modules are free. Hence, the object E can be represented by some matrix factorization of the following form

$$\left(R^{\oplus n_1} \xrightarrow{\varphi_1} R^{\oplus n_0} \xrightarrow{\varphi_0} R^{\oplus n_1} \right).$$

If $E \otimes_R R/\mathfrak{m} = 0$, there exist homotopies $h_0 : (R/\mathfrak{m})^{\oplus n_0} \rightarrow (R/\mathfrak{m})^{\oplus n_1}$ and $h_1 : (R/\mathfrak{m})^{\oplus n_1} \rightarrow (R/\mathfrak{m})^{\oplus n_0}$ such that $\mathrm{id}_{(R/\mathfrak{m})^{\oplus n_0}} = (\varphi_1 \otimes R/\mathfrak{m})h_0 + h_1(\varphi_0 \otimes R/\mathfrak{m})$ and $\mathrm{id}_{(R/\mathfrak{m})^{\oplus n_1}} = (\varphi_0 \otimes R/\mathfrak{m})h_1 + h_0(\varphi_1 \otimes R/\mathfrak{m})$. Since h_0 and h_1 can be represented by a matrix of units in R , there exist homomorphisms $\overline{h}_0 : R^{\oplus n_0} \rightarrow R^{\oplus n_1}$ and $\overline{h}_1 : R^{\oplus n_1} \rightarrow R^{\oplus n_0}$ such that $\overline{h}_i \otimes_R R/\mathfrak{m} = h_i$ for $i = 0, 1$. Set

$$\begin{aligned} \alpha_1 &:= \varphi_0 \overline{h}_1 + \overline{h}_0 \varphi_1 : R^{\oplus n_1} \rightarrow R^{\oplus n_1} \\ \alpha_0 &:= \varphi_1 \overline{h}_0 + \overline{h}_1 \varphi_0 : R^{\oplus n_0} \rightarrow R^{\oplus n_0}. \end{aligned}$$

Then the pair $\alpha := (\alpha_1, \alpha_0)$ defines an endomorphism of E in the exact category $\mathrm{MF}(R, w)$. By construction, $\alpha = 0$ in $\mathrm{KMF}(R, w)$. To show that $E = 0$ in $\mathrm{KMF}(R, w)$, it is enough to show that $\alpha : E \rightarrow E$ is an automorphism in $\mathrm{MF}(R, w)$. For each $i \in \{0, 1\}$, we only need to show that α_i is an automorphism. Since the tensor product $(-) \otimes_R R/\mathfrak{m}$ is a right exact functor and $\alpha_i \otimes_R R/\mathfrak{m} = \mathrm{id}$, we have

$$\mathrm{Cok}(\alpha_i) \otimes_R R/\mathfrak{m} \cong \mathrm{Cok}(\alpha_i \otimes_R R/\mathfrak{m}) = 0.$$

By Nakayama's lemma, the above implies that $\text{Cok}(\alpha_i) = 0$. Hence α_i is an automorphism by [Mat2, Theorem 2.4]. \square

For later use, we provide the following lemma.

Lemma 2.23. *Let R be a ring, and let $F = \left(R \xrightarrow{f_1} R \xrightarrow{f_0} R\right) \in \text{KMF}(R, f_0 f_1)$ be an object. Then we have the following equality of subsets of $\text{Spec } R$:*

$$\text{Supp}(F) = \bigcap_{i=0,1} Z(f_i),$$

where $Z(f_i) := \{p \in \text{Spec } R \mid f_i \otimes k(p) = 0\}$.

Proof. (\subseteq) If $p \in \text{Supp}(F)$, then $F \otimes k(p) \neq 0$ in $\text{KMF}(k(p), (f_0 f_1) \otimes k(p))$ by Lemma 2.22. Suppose $p \notin Z(f_i)$ for some i . Then $f_i \otimes k(p)$ is a unit in $k(p)$, and hence $F \otimes k(p)$ is homotopic to zero. This contradicts to $F \otimes k(p) \neq 0$. Hence $p \in \bigcap_{i=0,1} Z(f_i)$.

(\supseteq) If $p \in \bigcap_{i=0,1} Z(f_i)$. Then $f_i \otimes k(p) = 0$ for $i = 0, 1$, and so $H_i(F \otimes k(p)) = k(p) \neq 0$. Hence by [LS, Proposition 2.30], $F \otimes k(p) \neq 0$. Again by Lemma 2.22, $p \in \text{Supp}(F)$. \square

The following lemma is useful to compute the support of tensor products of matrix factorizations.

Lemma 2.24. *Let $V, W \in \Gamma(X, L)$ be any global sections of L , and let $E \in \text{DMF}(X, L, V)$ and $F \in \text{DMF}(X, L, W)$ be objects. Let $p \in X$ be a point such that $V \otimes k(p) = W \otimes k(p) = 0$. Then $p \in \text{Supp}(E \otimes F)$ if and only if $p \in \text{Supp}(E) \cap \text{Supp}(F)$.*

Proof. We have $(E \otimes F) \otimes k(p) \cong (E \otimes k(p)) \otimes (F \otimes k(p))$ in $\text{KMF}(k(p), 0)$. By Lemma 2.22, it is enough to show that for a field k , and for objects $M, N \in \text{KMF}(k, 0)$, $M \otimes N \neq 0$ if and only if $M \neq 0$ and $N \neq 0$. By Lemma 2.16, for $i = 0, 1$, we may assume $\varphi_i^M = \varphi_i^N = 0$, and then $H_i(M) = M_i$ and $H_i(N) = N_i$. Then, since $\varphi_i^{M \otimes N} = 0$ for $i = 0, 1$, we have

$$\begin{aligned} H_1(M \otimes N) &= \left(H_1(M) \otimes H_0(N)\right) \oplus \left(H_0(M) \otimes H_1(N)\right) \\ H_0(M \otimes N) &= \left(H_0(M) \otimes H_0(N)\right) \oplus \left(H_1(M) \otimes H_1(N)\right). \end{aligned}$$

Hence, by [LS, Proposition 2.30], we see that $M \otimes N \neq 0$ if and only if $M \neq 0$ and $N \neq 0$. \square

At the end of this section, we organize fundamental properties of supports of matrix factorizations.

Lemma 2.25. *Let $E, F, G \in \text{DMF}(X, L, W)$ be objects. We have the following.*

- (1) $\text{Supp}(E \oplus F) = \text{Supp}(E) \cup \text{Supp}(F)$.
- (2) $\text{Supp}(F[1]) = \text{Supp}(F)$.
- (3) $\text{Supp}(E) \subseteq \text{Supp}(F) \cup \text{Supp}(G)$ for any distinguished triangle $E \rightarrow F \rightarrow G \rightarrow E[1]$.
- (4) $\text{Supp}(E \otimes F) = \text{Supp}(E) \cap \text{Supp}(F)$.

Proof. (1), (2), and (3) are obvious. If $p \in \text{Supp}(M)$ for some object $M \in \text{DMF}(X, L, W)$, then $W \otimes k(p) = 0$, since $\text{KMF}(k(p), W \otimes k(p)) = 0$ if $W \otimes k(p) \neq 0$. Hence (4) follows from Lemma 2.24. \square

3. RELATIVE SINGULAR LOCUS AND SINGULARITY CATEGORY

In this section, we define relative singular loci and prove some properties about it. Let S be a Noetherian scheme and let $F \in \text{D}^b(\text{coh } S)$ be a bounded complex of coherent sheaves. The complex F is called **perfect** if it is locally quasi-isomorphic to a bounded complex of locally free sheaves of finite rank. $\text{Perf } S \subset \text{D}^b(\text{coh } S)$ denotes the thick subcategory of perfect complexes.

We define globally/locally relative singular locus. Recall our notation $S_p := \text{Spec}(\mathcal{O}_{S,p})$ for any point $p \in S$.

Definition 3.1. Let S be a Noetherian scheme, and let $i : T \hookrightarrow S$ be a closed immersion.

- (1) The subset $\text{Sing}(T/S) \subset T$, called the **singular locus of T globally relative** (or just **relative**) **to S** , is defined by

$$\text{Sing}(T/S) := \{p \in T \mid \exists F \in \text{coh } T \text{ such that } F_p \notin \text{Perf } T_p \text{ and } i_*(F) \in \text{Perf } S\}$$

- (2) The subset $\text{Sing}^{\text{loc}}(T/S) \subset T$, called the **singular locus of T locally relative to S** , is defined by

$$\text{Sing}^{\text{loc}}(T/S) := \{p \in T \mid \exists F \in \text{coh } T_p \text{ such that } F \notin \text{Perf } T_p \text{ and } i_{p*}(F) \in \text{Perf } S_p\}$$

where $i_p : T_p \hookrightarrow S_p$ is the closed immersion induced by $i : T \hookrightarrow S$

Proposition 3.2. *Let S be a Noetherian scheme, and let $i : T \hookrightarrow S$ be a closed immersion. Then we have*

$$\text{Sing}(T/S) \subseteq \text{Sing}^{\text{loc}}(T/S) \subseteq \text{Sing}(T).$$

Furthermore, if S is regular, globally and locally relative singular loci coincide with usual singular locus;

$$\text{Sing}(T/S) = \text{Sing}^{\text{loc}}(T/S) = \text{Sing}(T).$$

Proof. The first assertion $\text{Sing}(T/S) \subseteq \text{Sing}^{\text{loc}}(T/S) \subseteq \text{Sing}(T)$ is obvious.

For the latter assertion, assume that S is regular. If $\text{Sing}(T) = \emptyset$, $\text{Sing}(T/S) = \text{Sing}^{\text{loc}}(T/S) = \text{Sing}(T) = \emptyset$ by the former assertion. Assume that $\text{Sing}(T) \neq \emptyset$, and let $p \in \text{Sing}(T)$ be a singular point. It is enough to show that $p \in \text{Sing}(T/S)$. Since the projective dimension, denoted by $\text{pd}_{\mathcal{O}_{T,p}} k(p)$, of $k(p)$ as $\mathcal{O}_{T,p}$ -module coincides with the global dimension of $\mathcal{O}_{T,p}$, we have $\text{pd}_{\mathcal{O}_{T,p}} k(p) = \infty$. This implies that $k(p) \notin \text{Perf } T_p$. Let $\{a_1, \dots, a_r\} \subset \mathcal{O}_{T,p}$ is a generator of the maximal ideal \mathfrak{m}_p of the local ring $\mathcal{O}_{T,p}$. Then there exist a small open affine neighborhood $U = \text{Spec } R \subset T$ of p and elements $b_1, \dots, b_r \in R$ such that $(b_i)_p = a_i$. Let $I := \langle b_1, \dots, b_r \rangle$ be the ideal of R generated by b_i . Then $I_p \cong \mathfrak{m}_p$ and $(R/I)_p \cong k(p)$. Take an extension $F \in \text{coh } T$ of the coherent sheaf $\widetilde{R/I} \in \text{coh } U$, i.e. $F|_U \cong \widetilde{R/I}$. Then $F_p \cong k(p) \notin \text{Perf } T_p$. Since S is regular, we have $\text{D}^b(\text{coh } S) = \text{Perf } S$, and so $i_*(F) \in \text{Perf } S$. Hence $p \in \text{Sing}(T/S)$. \square

The locally relative singular locus has a local property.

Lemma 3.3. *Let $i : T \hookrightarrow S$ be a closed immersion of Noetherian schemes. Then we have*

$$\text{Sing}^{\text{loc}}(T/S) = \bigcup_{p \in T} \text{Sing}^{\text{loc}}(T_p/S_p),$$

where the sets $\text{Sing}^{\text{loc}}(T_p/S_p)$ on the right hand side are considered as the subsets of T via the natural injective maps $j_p : T_p \hookrightarrow T$.

Proof. If $p \in \text{Sing}^{\text{loc}}(T/S)$, then $\mathfrak{m}_p \in \text{Sing}^{\text{loc}}(T_p/S_p)$, and $j_p(\mathfrak{m}_p) = p$. This means that $p \in \text{Sing}^{\text{loc}}(T_p/S_p)$, and so $\text{Sing}^{\text{loc}}(T/S) \subseteq \bigcup_{p \in T} \text{Sing}^{\text{loc}}(T_p/S_p)$. If $q \in \text{Sing}^{\text{loc}}(T_p/S_p)$ for some $p \in T$, then, since $(T_p)_q \cong T_{j_p(q)}$, $j_p(q) \in \text{Sing}^{\text{loc}}(T/S)$. Hence $\bigcup_{p \in T} \text{Sing}^{\text{loc}}(T_p/S_p) \subseteq \text{Sing}^{\text{loc}}(T/S)$. \square

Next we recall singularity categories. Let X be a separated Noetherian scheme with *resolution property*, i.e. for any $F \in \text{coh } X$, there exist a locally free coherent sheaf E and a surjective homomorphism $E \twoheadrightarrow F$. Following [Orl1], we define the triangulated category of singularities $\text{D}^{\text{sg}}(X)$ as the Verdier quotient

$$\text{D}^{\text{sg}}(X) = \text{D}^b(\text{coh } X) / \text{Perf } X.$$

In our assumption, $\text{Perf } X$ coincides with thick subcategory of complexes which are quasi-isomorphic to a bounded complex of locally free sheaves of finite rank.

We recall that derived matrix factorization categories can be embedded into singularity categories. Let L be a line bundle on X , and $W \in \Gamma(X, L)$ be a non-zero-divisor, i.e. the induced homomorphism $W : \mathcal{O}_X \rightarrow L$ is injective, and denote by X_0 the zero scheme of W . Denote by $j : X_0 \hookrightarrow X$ the closed immersion. Since the direct image $j_* : \text{D}^b(\text{coh } X_0) \rightarrow \text{D}^b(\text{coh } X)$ preserves perfect complexes by [TT, Proposition 2.7.(a)], it induces an exact functor

$$j_* : \text{D}^{\text{sg}}(X_0) \rightarrow \text{D}^{\text{sg}}(X).$$

As in [Orl2], the cokernel functor $\Sigma : \text{MF}(X, L, W) \rightarrow \text{coh } X_0$ defined by $\Sigma(F) := \text{Cok}(\varphi_1^F)$ induces an exact functor

$$\Sigma : \text{DMF}(X, L, W) \rightarrow \text{D}^{\text{sg}}(X_0).$$

Theorem 3.4 ([Orl2, Theorem 1], [EP, Theorem 2.7]). *The functor $\Sigma : \mathrm{DMF}(X, L, W) \rightarrow \mathrm{D}^{\mathrm{sg}}(X_0)$ is fully faithful, and the essential image of Σ is the thick subcategory consisting of objects F such that $j_*(F) = 0 \in \mathrm{D}^{\mathrm{sg}}(X)$. In particular, if X is regular, Σ is an equivalence.*

The following result is the key motivation for our definitions of relative singular loci.

Proposition 3.5. (1) *We have an equality of subsets of X*

$$\mathrm{Sing}(X_0/X) = \bigcup_{F \in \mathrm{DMF}(X, L, W)} \mathrm{Supp}(F).$$

(2) *We have an equality of subsets of X*

$$\mathrm{Sing}^{\mathrm{loc}}(X_0/X) = \{p \in X \mid \mathrm{KMF}(X_p, W_p) \neq 0\}$$

Proof. Since (2) follows from a similar proof of (1), we prove only (1).

If $p \in \mathrm{Sing}(X_0/X)$, by definition, there exists $A \in \mathrm{D}^{\mathrm{sg}}(X_0)$ such that $A_p \neq 0$ in $\mathrm{D}^{\mathrm{sg}}((X_0)_p)$ and $j_*(A) = 0$ in $\mathrm{D}^{\mathrm{sg}}(X)$. Then by Theorem 3.4, there is an object $F \in \mathrm{DMF}(X, L, W)$ such that $\Sigma(F) \cong A$. Then we have $A_p \cong \Sigma(F)_p \cong \Sigma_p(F_p)$, where $\Sigma_p : \mathrm{KMF}(X_p, W_p) \rightarrow \mathrm{D}^{\mathrm{sg}}((X_0)_p)$ is the exact functor defined as above. Since $A_p \neq 0$ and Σ_p is fully faithful, $F_p \neq 0$ and hence $p \in \mathrm{Supp}(F)$.

Conversely, if $p \in \mathrm{Supp}(F)$ for some $F \in \mathrm{DMF}(X, L, W)$, then $F_p \neq 0 \in \mathrm{KMF}(X_p, W_p)$. Since $\Sigma_p : \mathrm{KMF}(X_p, W_p) \rightarrow \mathrm{D}^{\mathrm{sg}}((X_0)_p)$ is fully faithful, $\Sigma(F)_p \cong \Sigma_p(F_p) \neq 0$, and so $\Sigma(F)_p \notin \mathrm{Perf}(X_0)_p$. Furthermore, $j_*(\Sigma(F)) \in \mathrm{Perf} X$ by Theorem 3.4. Hence $p \in \mathrm{Sing}(X_0/X)$. \square

Lemma 3.6. *Let (R, \mathfrak{m}) be a local ring, and let $W \in R$ be an element. The category $\mathrm{KMF}(R, W)$ has a non-zero object if and only if $W \in \mathfrak{m}^2$.*

Proof. Assume that $W \in \mathfrak{m}^2$. Then there exist non-units $m_i, n_i \in \mathfrak{m}$ for $1 \leq i \leq r$ such that $W = \sum_{i=1}^r m_i n_i$. Set $K_i := (R \xrightarrow{m_i} R \xrightarrow{n_i} R) \in \mathrm{KMF}(R, m_i n_i)$ and $\mathcal{K} := \bigotimes_{i=1}^r K_i \in \mathrm{KMF}(R, W)$. Then we claim that $\mathcal{K} \neq 0$ in $\mathrm{KMF}(R, W)$. Indeed, if $\mathcal{K} = 0$ in $\mathrm{KMF}(R, W)$, there are morphisms $h_0 : \mathcal{K}_0 \rightarrow \mathcal{K}_1$, $h_1 : \mathcal{K}_1 \rightarrow \mathcal{K}_0$ such that $\mathrm{id}_{\mathcal{K}_0} = \varphi_1^{\mathcal{K}} h_0 + h_1 \varphi_0^{\mathcal{K}}$. Since each $\varphi_i^{\mathcal{K}}$ is a matrix whose entries are non-units in R , the equation implies that $1_R \in \mathfrak{m}$, which is a contradiction.

For the converse, let $F \in \mathrm{KMF}(R, W)$ is a non-zero object. Since R is local, every locally free modules are free modules. Hence each φ_i^F is a r -square matrix $(f_{m,n}^i)_{1 \leq m,n \leq r}$ in elements in R . We claim that F is isomorphic to a matrix factorization F' such that all entries of matrices $\varphi_i^{F'}$ are non-units. Indeed, if there is a unit entry $u \in R$ in the square matrix $\varphi_1^F = (f_{m,n}^1)_{1 \leq m,n \leq r}$, applying elementary row/column operations, we may assume that $u = f_{1,1}^1$ and $f_{m,1}^1 = f_{1,m}^1 = 0$ for $m \neq 1$. Then we see that $f_{m,1}^0 = f_{1,m}^0 = 0$ for $m \neq 1$. Hence the object $(R \xrightarrow{u} R \xrightarrow{u^{-1}W} R)$ is a direct summand of F , but this is isomorphic to the zero object in $\mathrm{KMF}(R, W)$. Since the rank of matrices φ_i^F are finite and $F \neq 0$, repeating this process, we may assume that all entries of φ_i^F are non-units, and hence $W \in \mathfrak{m}^2$ since $W = \varphi_0^F \varphi_1^F$. \square

The following result is useful to compute the relative singular loci of zero schemes of regular sections of line bundles.

Proposition 3.7. *Notation is same as above. We have*

$$\mathrm{Sing}^{\mathrm{loc}}(X_0/X) = \{p \in X \mid W_p \in \mathfrak{m}_p^2\}$$

Proof. This follows from Proposition 3.5.(2) and Lemma 3.6. \square

Remark 3.8. Recall that the **critical locus** $\mathrm{Crit}(\varphi)$ of a function $\varphi \in \Gamma(Y, \mathcal{O}_Y)$ on a complex analytic space (Y, \mathcal{O}_Y) is defined by

$$\mathrm{Crit}(\varphi) := \{p \in Y \mid \varphi_p - \varphi(p) \in \mathfrak{m}_p^2\},$$

where \mathfrak{m}_p is the maximal ideal of the local ring $\mathcal{O}_{Y,p}$. Assume that X is a quasi-projective variety over \mathbb{C} , and let $f \in \Gamma(X, \mathcal{O}_X)$ be a regular function which is a non-zero-divisor. Denote by $(X^{\mathrm{an}}, \mathcal{O}_X^{\mathrm{an}})$ the complex analytic space associated to X , and let $f^{\mathrm{an}} \in \Gamma(X^{\mathrm{an}}, \mathcal{O}_X^{\mathrm{an}})$ is the function associated to f . For any $p \in X^{\mathrm{an}}$, since the morphism of local rings $\varphi_p : \mathcal{O}_{X,p} \rightarrow \mathcal{O}_{X,p}^{\mathrm{an}}$ is flat [Ser],

φ_p induces the isomorphism $\mathfrak{m}_p/\mathfrak{m}_p^2 \cong \mathfrak{m}_p^{\text{an}}/(\mathfrak{m}_p^{\text{an}})^2$ (see [Mat1, Theorem 49]), and so $f_p \in \mathfrak{m}_p^2$ if and only if $f_p^{\text{an}} \in (\mathfrak{m}_p^{\text{an}})^2$. Hence Proposition 3.7 implies the following equality of sets;

$$\text{Sing}^{\text{loc}}(f^{-1}(0)/X) \cap X^{\text{an}} = \text{Crit}(f^{\text{an}}) \cap \text{Zero}(f^{\text{an}}),$$

where $\text{Zero}(f^{\text{an}})$ is the zero locus of f^{an} .

Recall that the **codimension** of a Noetherian local ring (R, \mathfrak{m}) is defined by

$$\text{codim}(R) := \text{emb. dim}(R) - \dim(R),$$

where $\text{emb. dim}(R) := \dim_{R/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2)$ is the dimension of R/\mathfrak{m} -vector space of $\mathfrak{m}/\mathfrak{m}^2$, which is called the **embedding dimension** of (R, \mathfrak{m}) . The following result provides a numerical characterization of $\text{Sing}(X_0/X)$.

Proposition 3.9. *Assume that for any $p \in X_0$ we have $\dim(\mathcal{O}_{X,p}) - \dim(\mathcal{O}_{X_0,p}) = 1$. Then we have the following equality of sets;*

$$\text{Sing}^{\text{loc}}(X_0/X) = \{p \in X_0 \mid \text{codim}(\mathcal{O}_{X_0,p}) > \text{codim}(\mathcal{O}_{X,p})\}.$$

Proof. Let $p \in X_0$ be a point, and denote by $\mathfrak{m}_p \subset \mathcal{O}_{X,p}$ and $\mathfrak{n}_p \subset \mathcal{O}_{X_0,p}$ the maximal ideals of $\mathcal{O}_{X,p}$ and $\mathcal{O}_{X_0,p}$ respectively. The surjective homomorphism $\pi : \mathcal{O}_{X,p} \rightarrow \mathcal{O}_{X_0,p}$ induces a surjective map

$$\bar{\pi} : \mathfrak{m}_p/\mathfrak{m}_p^2 \rightarrow \mathfrak{n}_p/\mathfrak{n}_p^2.$$

Hence we have

$$\text{emb. dim}(\mathcal{O}_{X_0,p}) \leq \text{emb. dim}(\mathcal{O}_{X,p}).$$

Moreover, since $\dim(\mathcal{O}_{X,p}) - \dim(\mathcal{O}_{X_0,p}) = 1$, we have $\text{codim}(\mathcal{O}_{X_0,p}) > \text{codim}(\mathcal{O}_{X,p})$ if and only if $\text{emb. dim}(\mathcal{O}_{X_0,p}) \geq \text{emb. dim}(\mathcal{O}_{X,p})$. Hence, by Proposition 3.7, it is enough to show that

$$W_p \in \mathfrak{m}_p^2 \Leftrightarrow \bar{\pi} \text{ is injective.}$$

For (\Rightarrow) , assume that $W_p \in \mathfrak{m}_p^2$, and let $x \in \mathfrak{m}_p$ with $\pi(x) \in \mathfrak{n}_p^2$. Then, since $\mathcal{O}_{X_0,p} \cong \mathcal{O}_{X,p}/\langle W_p \rangle$, there exists $y \in \mathcal{O}_{X,p}$ such that

$$x + yW_p \in \mathfrak{m}_p^2.$$

Hence $x \in \mathfrak{m}_p^2$ by the assumption $W_p \in \mathfrak{m}_p^2$. This means that $\bar{\pi}$ is injective. For (\Leftarrow) , assume that $\bar{\pi}$ is injective. Since $p \in X_0$, $W_p \in \mathfrak{m}_p$. Since $\pi(W_p) = 0$ and $\bar{\pi}$ is injective, we have $W_p \in \mathfrak{m}_p^2$. \square

Next, we show some properties describing relationships between relative singular loci and locally relative singular loci.

Proposition 3.10. *Let R be a Noetherian ring, and let $W \in R$ is a non-zero-divisor. Set $X := \text{Spec } R$ and $X_0 := \text{Spec}(R/W)$.*

(1) *We have*

$$\text{Sing}(X_0/X) \cap \text{Max}(R/W) = \text{Sing}^{\text{loc}}(X_0/X) \cap \text{Max}(R/W).$$

(2) *Let $p \in X_0$ be a point with $p \notin \text{Sing}^{\text{loc}}(X_0/X)$. Let $q \in X_0$ be a point and denote by $\overline{\{q\}}$ the closure of q . If $p \in \overline{\{q\}}$, then $q \notin \text{Sing}(X_0/X)$.*

Proof. (1) The inclusion $\text{Sing}(X_0/X) \cap \text{Max}(R/W) \subseteq \text{Sing}^{\text{loc}}(X_0/X) \cap \text{Max}(R/W)$ follows from Proposition 3.2. To show the opposite inclusion, let $\mathfrak{m} \in \text{Sing}^{\text{loc}}(X_0/X) \cap \text{Max}(R/W)$ be a maximal ideal of R/W which is contained in $\text{Sing}^{\text{loc}}(X_0/X)$, and let $\bar{\mathfrak{m}} \in \text{Max } R$ be the maximal ideal of R such that $i(\mathfrak{m}) = \bar{\mathfrak{m}}$, where $i : X_0 \hookrightarrow X$ is the closed immersion. By Proposition 3.7, there exist elements $m_i, n_i, r \in R$ ($1 \leq i \leq r$) such that $m_i \in \bar{\mathfrak{m}}$, $n_i \in \bar{\mathfrak{m}}$, $r \notin \bar{\mathfrak{m}}$, and $rW = \sum_{i=1}^r m_i n_i$. Since $r \notin \bar{\mathfrak{m}}$ and $\bar{\mathfrak{m}}$ is maximal, we have $\langle r \rangle + \bar{\mathfrak{m}} = R$, and so there exists an element $a \in R$ such that $1 - ar \in \bar{\mathfrak{m}}$. Then we have

$$W = (1 - ar)W + \sum_{i=1}^r am_i n_i.$$

Consider the following matrix factorizations

$$\begin{aligned} K_0 &:= \left(R \xrightarrow{W} R \xrightarrow{1-ar} R \right) \in \text{KMF}(R, (1-ar)W) \\ K_i &:= \left(R \xrightarrow{am_i} R \xrightarrow{n_i} R \right) \in \text{KMF}(R, am_i n_i) \\ \mathcal{K} &:= \bigotimes_{i=0}^r K_i \in \text{KMF}(R, W). \end{aligned}$$

Then, by Lemma 2.23 and Lemma 2.24, we see that $\overline{\mathfrak{m}} \in \bigcap_{i=0}^r \text{Supp}(K_i) = \text{Supp}(\mathcal{K})$. Proposition 3.5.(1) implies that $\mathfrak{m} \in \text{Sing}(X_0/X)$.

(2) Assume that $q \in \text{Sing}(X_0/X)$. Since the relative singular locus $\text{Sing}(X_0/X)$ is a union of closed subsets by Proposition 3.5.(1), we have $\overline{\{q\}} \subseteq \text{Sing}(X_0/X)$. Since $p \in \overline{\{q\}}$ and $\text{Sing}(X_0/X) \subseteq \text{Sing}^{\text{loc}}(X_0/X)$, $p \in \text{Sing}^{\text{loc}}(X_0/X)$, which contradicts to the assumption of p . \square

At the end of this section, we compute examples of relative singular loci using the above results.

Example 3.11. We give two examples of the relative singular loci which are not equal to the usual singular loci.

(1) Let $R := \mathbb{C}[x, y]/\langle x^n \rangle$ for $n > 1$, and let $W := \overline{y} \in R$. Set $X := \text{Spec } R$ and $X_0 := \text{Spec}(R/W)$. Although $\text{Sing}(X_0) = \{\text{pt}\} \neq \emptyset$, by Proposition 3.7, we have

$$\text{Sing}(X_0/X) = \text{Sing}^{\text{loc}}(X_0/X) = \emptyset.$$

(2) Let $R := \mathbb{C}[x, y, z, w]/\langle xy - zw \rangle$, and let $W := \overline{w} \in R$. Set $X := \text{Spec } R$ and $X_0 := \text{Spec}(R/W)$. Then we have

$$\text{Sing}(X_0) = \{ \langle \overline{x}, \overline{y} \rangle, \langle \overline{x}, \overline{y}, \overline{z} - a \rangle \mid a \in \mathbb{C} \}.$$

By Proposition 3.7, we see that

$$\text{Sing}^{\text{loc}}(X_0/X) = \text{Sing}(X_0) \setminus \{ \langle \overline{x}, \overline{y}, \overline{z} \rangle \},$$

and, by Proposition 3.10, we have

$$\text{Sing}(X_0/X) = \text{Sing}^{\text{loc}}(X_0/X) \setminus \{ \langle \overline{x}, \overline{y} \rangle \}.$$

In this example, all kinds of singular loci are different;

$$\text{Sing}(X_0/X) \subsetneq \text{Sing}^{\text{loc}}(X_0/X) \subsetneq \text{Sing}(X_0).$$

4. TENSOR NILPOTENCE PROPERTIES

In this section, we prove the tensor nilpotent properties, which will be necessary for our main result. The properties are analogous to [Tho, Theorem 3.6, 3.8], and the strategy of the proof is similar to loc. cit. Let X be a Noetherian scheme, and let $W \in \Gamma(X, L)$ be a global section of a line bundle L on X .

4.1. Mayer-Vietoris sequence. We provide a Mayer-Vietoris sequence for factorizations for the proof of the tensor nilpotence properties in the next section. For an open immersion $i : U \hookrightarrow X$, consider an induced LG model $(U, L|_U, W|_U)$. Let $i_! : \text{Mod } \mathcal{O}_U \rightarrow \text{Mod } \mathcal{O}_X$ be the extension by zero. For an object $F = \left(F_1 \xrightarrow{\varphi_1^F} F_0 \xrightarrow{\varphi_0^F} F_1 \otimes L|_U \right) \in \text{Sh}(U, L|_U, W|_U)$, we define an object $i_!(F) \in \text{Sh}(X, L, W)$ by

$$i_!(F) := \left(i_!(F_1) \xrightarrow{i_!(\varphi_1^F)} i_!(F_0) \xrightarrow{\sigma \circ i_!(\varphi_0^F)} i_!(F_1) \otimes L \right),$$

where $\sigma : i_!(F_1 \otimes L|_U) \xrightarrow{\sim} i_!(F_1) \otimes L$ is a natural isomorphism. This defines an exact functor

$$i_! : \text{Sh}(U, L|_U, W|_U) \rightarrow \text{Sh}(X, L, W).$$

Similarly, the inverse image functor $i^* : \text{Mod } \mathcal{O}_X \rightarrow \text{Mod } \mathcal{O}_U$ defines an exact functor

$$i^* : \text{Sh}(X, L, W) \rightarrow \text{Sh}(U, L|_U, W|_U).$$

These functors induce an exact functors between homotopy categories;

$$i_! : \text{KSh}(U, L|_U, W|_U) \rightarrow \text{KSh}(X, L, W)$$

Step 2. By the above step, we may assume that $X = \operatorname{Spec} R$ is affine and L is an invertible R -module. Recall that $\operatorname{DMF}(R, L, W) \cong \operatorname{KMF}(R, L, W)$ by Proposition 2.12. In this step, we reduce to the case when $W = 0$ and $E = (0 \rightarrow R \rightarrow 0)$. We have the following adjunction

$$\Phi : \operatorname{Hom}_{\operatorname{KMF}(R, L, W)}(E, F) \xrightarrow{\sim} \operatorname{Hom}_{\operatorname{KMF}(R, L, 0)}(R, E^\vee \otimes F).$$

Via natural isomorphisms, we can identify $\Phi(f^{\otimes n})$ with $\Phi(f)^{\otimes n}$ and $\Phi(f \otimes k(p))$ with $\Phi(f^{\otimes n}) \otimes k(p)$ respectively. Therefore, we may assume that $W = 0$ and $E = R$.

Step 3. Since the components F_i of F and L are locally free, by the above first step, shrinking $X = \operatorname{Spec} R$ if necessary, we may assume that $X = \operatorname{Spec} R$ is affine scheme such that each F_i is free R -module and $L \cong R$. Furthermore, by the second step, we may assume that $E = R$ and $W = 0$. Hence, since the natural isomorphism $\operatorname{Hom}_{\operatorname{MF}(R, 0)}(R, F) \cong \operatorname{Ker}(\varphi_0^F)$ induces the isomorphism $\operatorname{Hom}_{\operatorname{KMF}(R, 0)}(R, F) \cong H_0(F)$, we only need to show that if $f \in H_0(F)$ satisfies $f \otimes k(p) = 0 \in H_0(F \otimes k(p))$ for any $p \in \operatorname{Spec} R$, then $f^{\otimes n} = 0 \in H_0(F^{\otimes n})$ for some $n > 0$.

In this step, we reduce to the case when the Noetherian ring R is of finite Krull dimension. Since the components F_i of F are free R -modules, the morphisms φ_i^F can be represented by a matrices whose entries are elements in R . Let $\{R_\alpha\}_{\alpha \in \mathcal{A}}$ be the family of all subrings $R_\alpha \subset R$ of R such that $\dim R_\alpha < \infty$ and R_α contains all entries of matrices φ_1 and φ_0 . Then for any $\alpha \in \mathcal{A}$, there is the natural object $F_\alpha \in \operatorname{KMF}(R_\alpha, 0)$ such that its components $(F_\alpha)_i$ are free R_α -modules and $\pi_\alpha^*(F_\alpha) \cong F$ in the additive category $\operatorname{MF}(R, 0)$, where $\pi_\alpha : \operatorname{Spec} R \rightarrow \operatorname{Spec} R_\alpha$ is the morphism induced by the inclusion $R_\alpha \subset R$. Let

$$F^\bullet := (\cdots 0 \rightarrow F_1 \xrightarrow{\varphi_1^F} F_0 \xrightarrow{\varphi_0^F} F_1 \rightarrow 0 \cdots)$$

$$F_\alpha^\bullet := (\cdots 0 \rightarrow (F_\alpha)_1 \xrightarrow{\varphi_1^{F_\alpha}} (F_\alpha)_0 \xrightarrow{\varphi_0^{F_\alpha}} (F_\alpha)_1 \rightarrow 0 \cdots)$$

be the complexes of free modules such that the term F_0 and $(F_\alpha)_0$ are of degree 0. Then we have $H^0(F^\bullet) = H_0(F)$ and $H^0(F^\bullet \otimes k(p)) = H_0(F \otimes k(p))$ for any $p \in \operatorname{Spec} R$. Since R is the direct colimit of the system $\{R_\alpha\}_{\alpha \in \mathcal{A}}$; $R = \varinjlim R_\alpha$, by the same argument as in the step (3.6.4) in the proof of [Tho, Theorem 3.6], if $f \otimes k(p) = 0$ in $H_0(F \otimes k(p))$ for any $p \in X$, there exist $\beta \in \mathcal{A}$ and an element $f_\beta \in H^0(F_\beta^\bullet) = H_0(F_\beta)$ such that $\pi_\beta^*(f_\beta) = f$ and $f_\beta \otimes k(p) = 0$ in $H^0(F_\beta^\bullet \otimes k(p)) = H_0(F_\beta \otimes k(p))$ for any $p \in \operatorname{Spec} R_\beta$. Therefore, if the assertion is true for $X = \operatorname{Spec} R_\beta$, there exists $n > 0$ such that $f_\beta^{\otimes n} = 0$ in $H_0(F_\beta^{\otimes n})$, and then $f^{\otimes n} = \pi_\beta^*(f_\beta^{\otimes n}) = 0$ in $H_0(F^{\otimes n})$. This completes the reduction to the case when the ring R is of finite Krull dimension.

Step 4. In this step, we reduce to the case when the Noetherian ring R of finite Krull dimension is reduced. Let $\mathfrak{N} \subset R$ be the ideal of nilpotent elements in R , and denote by $h : \operatorname{Spec} R/\mathfrak{N} \rightarrow \operatorname{Spec} R$ be the closed immersion. If $f \otimes k(p) = 0$ for any $p \in \operatorname{Spec} R$, then $h^*(f) \otimes k(q) = 0$ for any $q \in \operatorname{Spec} R/\mathfrak{N}$. Hence, for the reduction, we claim that, if $h^*(f)^{\otimes n'} = 0 \in H_0((h^*F)^{\otimes n'})$ for some $n' > 0$, then there exists $n > 0$ such that $f^{\otimes n} = 0 \in H_0(F^{\otimes n})$. Assume that $h^*(f)^{\otimes n'} = 0 \in H_0((h^*F)^{\otimes n'})$. Since $(h^*F)^{\otimes n'} \cong h^*(F^{\otimes n'}) = F^{\otimes n'} \otimes_R R/\mathfrak{N}$, the assumption implies that there exist elements $x \in (F^{\otimes n'})_1$ and $y \in \mathfrak{N}((F^{\otimes n'})_0) \subset (F^{\otimes n'})_0$ such that $f^{\otimes n'} = \varphi_1^{F^{\otimes n'}}(x) + y$ in $(F^{\otimes n'})_0$. Since $y \in \mathfrak{N}((F^{\otimes n'})_0)$, there is a positive integer m such that $y^{\otimes m} = 0$ in the free R -module $((F^{\otimes n'})_0)^{\otimes m}$. Therefore, it is enough to show the following claim:

Let S be a ring, and let $E \in \operatorname{MF}(S, 0)$ be an object such that its components E_i are free S -modules. For $e \in \operatorname{Ker}(\varphi_0^E) \subset E_0$, suppose that there are elements $u \in E_1$ and $v \in E_0$ such that $e = \varphi_1^E(u) + v$ and $v^{\otimes n} = 0$ in the S -free module $(E_0)^{\otimes n}$ for some $n > 0$. Then, considering $e^{\otimes n}$ as an element in $(E^{\otimes n})_0$ via the natural split mono $(E_0)^{\otimes n} \hookrightarrow (E^{\otimes n})_0$, there is an element $w \in (E^{\otimes n})_1$ such that $\varphi_1^{E^{\otimes n}}(w) = e^{\otimes n}$ in $(E^{\otimes n})_0$. In particular, $e^{\otimes n} = 0 \in H_0(E^{\otimes n})$.

The element $e^{\otimes n} = (\varphi_1^E(u) + v)^{\otimes n} \in (E_0)^{\otimes n}$ can be decomposed into the following form

$$e^{\otimes n} = \varphi_1^E(u) \otimes w_{n-1} + v \otimes \varphi_1^E(u) \otimes w_{n-2} + \cdots + v^{\otimes n-2} \otimes \varphi_1^E(u) \otimes w_1 + v^{\otimes n-1} \otimes \varphi_1^E(u),$$

where w_i is an element in $(E_0)^{\otimes i}$. For an ordered sequence (i_1, i_2, \dots, i_n) of $i_k \in \{0, 1\}$, set $E_{(i_1, i_2, \dots, i_n)} := E_{i_1} \otimes E_{i_2} \otimes \cdots \otimes E_{i_n}$, and set

$$\tilde{E} := E_{(1, 0, 0, \dots, 0)} \oplus E_{(0, 1, 0, \dots, 0)} \oplus \cdots \oplus E_{(0, \dots, 0, 1)}.$$

Let $\tilde{w} := (u \otimes w_{n-1}) \oplus (v \otimes u \otimes w_{n-2}) \oplus \cdots \oplus (v^{\otimes n-2} \otimes u \otimes w_1) \oplus (v^{\otimes n-1} \otimes u) \in \tilde{E}$, and let $w := \iota(\tilde{w}) \in (E^{\otimes n})_1$ be the image of \tilde{w} under the natural split mono $\iota : \tilde{E} \hookrightarrow (E^{\otimes n})_1$. Since $\varphi_0^E(\varphi_1^E(u)) = 0$, $\varphi_0^E(v) = \varphi_0^E(e - \varphi_1^E(u)) = 0$, and each $w_i \in (E_0)^{\otimes i}$ is a summation of elements of the form $a_1 \otimes a_2 \otimes \cdots \otimes a_i$ where a_k are either $\varphi_1^E(u)$ or v , we obtain an equality $\varphi_1^{E^{\otimes n}}(w) = e^{\otimes n}$ in $(E^{\otimes n})_0$. This completes the proof of the claim.

Step 5. Now we may assume that $X = \text{Spec } R$ is reduced affine scheme of finite Krull dimension, the components F_i of F are free R -modules, $L \cong R$, $W = 0$, and $E = R \in \text{KMF}(R, 0)$. In this step, we finish the proof by induction on the Krull dimension $d := \dim R$ of R .

If $d = 0$, then $R \cong \bigoplus_{p \in X} k(p)$ and $H_0(F) \cong \bigoplus_{p \in X} H_0(F \otimes k(p))$. Hence, if $f \otimes k(p) = 0 \in H_0(F \otimes k(p))$ for any $p \in X$, then $f = 0$ in $H_0(F)$.

Consider a case when $d > 0$, and assume that the result holds for Noetherian rings of dimension less than d . Denote by $\text{Min } R$ the finite set of all prime ideals of R of height zero. Then the product $\prod_{p \in \text{Min } R} k(p)$ of residue fields is isomorphic to the localization $S^{-1}R$ for the set S of all non zero-divisors in R , as the residue fields $k(p)$ is equal to the local ring R_p for any $p \in \text{Min } R$. By hypothesis, $f \otimes k(p) = 0 \in H_0(F \otimes k(p))$ for any $p \in \text{Min } R$, hence $f \otimes_R S^{-1}R = 0 \in H_0(F \otimes_R S^{-1}R)$. This means that, for a representative $f \in F_0$ of the equivalence class $f \in H_0(F)$, there exist elements $y \in F_1$ and $s \in S$ such that $sf = \varphi_1^F(y)$ in F_0 . Set $K_s := (R \xrightarrow{s} R \xrightarrow{0} R) \in \text{MF}(R, 0)$ and let $\gamma := (y, f) : K_s \rightarrow F$ be the morphism defined as the pair of morphisms $y : R \rightarrow F_1$ and $f : R \rightarrow F_0$ of R -modules. Denote by $i : \text{Spec } R/s \rightarrow \text{Spec } R$ the natural closed immersion. The canonical quotient $R \rightarrow i_*(R/s)$ naturally defines morphisms $\delta : K_s \rightarrow (0 \rightarrow i_*(R/s) \rightarrow 0)$ and $\alpha : (0 \rightarrow R \rightarrow 0) \rightarrow (0 \rightarrow i_*(R/s) \rightarrow 0)$ in $\text{coh}(R, 0)$. Since $s : R \rightarrow R$ is injective, we have an exact sequence

$$0 \rightarrow \left(R \xrightarrow{\text{id}} R \xrightarrow{0} R \right) \xrightarrow{(\text{id}, s)} K_s \xrightarrow{\delta} (0 \rightarrow i_*(R/s) \rightarrow 0) \rightarrow 0$$

in $\text{coh}(R, 0)$. Since $(R \xrightarrow{\text{id}} R \xrightarrow{0} R)$ is zero in $\text{Dcoh}(R, 0)$, δ is an isomorphism in $\text{Dcoh}(R, 0)$ by [LS, Lemma 2.7.(a)]. Set $\beta : (0 \rightarrow i_*(R/s) \rightarrow 0) \rightarrow F$ be the composition $\gamma \circ \delta^{-1}$ in $\text{Dcoh}(R, 0)$. Then the composition $\beta \circ \alpha : (0 \rightarrow R \rightarrow 0) \rightarrow F$ is equal to $f : (0 \rightarrow R \rightarrow 0) \rightarrow F$ in $\text{Dcoh}(R, 0)$, since $\alpha = \delta \circ \iota$ and $f = \delta \circ \iota$ in $\text{coh}(R, 0)$, where $\iota : (0 \rightarrow R \rightarrow 0) \rightarrow K_s$ is the morphism such that $\iota_1 = 0$ and $\iota_0 = \text{id}_R$. Hence, for any $n > 0$, we have the following commutative diagram in $\text{Dcoh}(R, 0)$:

$$\begin{array}{ccccccc} R & \xrightarrow{\sim} & R^{\otimes n} \otimes R & \xrightarrow{1 \otimes \alpha} & R^{\otimes n} \otimes i_*(R/s) & \xrightarrow{\sim} & i_*(R/s^{\otimes n}) \\ \downarrow f^{\otimes n+1} & & \downarrow f^{\otimes n} \otimes f & \searrow f^{\otimes n} \otimes 1 & \downarrow f^{\otimes n} \otimes 1 & & \downarrow i_*(i^* f^{\otimes n}) \\ & & & F^{\otimes n} \otimes R & & & \\ & & \swarrow 1 \otimes f & & \searrow 1 \otimes \alpha & & \\ F^{\otimes n+1} & \xrightarrow{\sim} & F^{\otimes n} \otimes F & \xleftarrow{1 \otimes \beta} & F^{\otimes n} \otimes i_*(R/s) & \xrightarrow{\sim} & i_*(i^* F^{\otimes n}) \end{array}$$

where $R = (0 \rightarrow R \rightarrow 0)$ and $i_*(R/s) = (0 \rightarrow i_*(R/s) \rightarrow 0)$ by abuse of notation. Since $\dim R/s < \dim R$ and $i^* f \otimes_{R/s} k(p) = f \otimes_R k(p) = 0$ in $H_0(i^* F \otimes k(p))$, by the induction hypothesis, there exists $m > 0$ such that $i^* f^{\otimes m} = 0$ in $\text{KMF}(R, 0)$, in particular $i^* f^{\otimes m} = 0$ in $\text{Dcoh}(R, 0)$. By the above commutative diagram, we see that $f^{\otimes m+1}$ factors through $i_*(i^* f^{\otimes m})$ in $\text{Dcoh}(R, 0)$, and hence $f^{\otimes m+1} = 0$ in $\text{Dcoh}(R, 0)$. Then, by Lemma 2.14.(5), $f^{\otimes m+1} = 0$ in $\text{KMF}(R, 0)$, and this completes the proof. \square

The following lemma is a consequence of the above lemma.

Lemma 4.3. *Let $a : E \rightarrow F$ be a morphism in $\text{DMF}(X, L, 0)$, and let $G \in \text{DMF}(X, L, W)$ be an object. If $a \otimes k(p) = 0$ in $\text{DMF}(k(p), 0)$ for all $p \in \text{Supp}(G)$, then there is an integer $n > 0$ such that $G \otimes (a^{\otimes n}) = 0$ in $\text{DMF}(X, L, W)$.*

Proof. Since $(G \otimes a) \otimes k(p) = 0$ in $\text{DMF}(k(p), W \otimes k(p))$ for any $p \in X$, by Lemma 4.2, there is a positive integer $n > 0$ such that $(G \otimes a)^{\otimes n} = G^{\otimes n} \otimes a^{\otimes n} = 0$ in $\text{DMF}(X, L, nW)$. Hence $((G^\vee)^{\otimes n-1} \otimes G^{\otimes n}) \otimes a^{\otimes n} = 0$, and so it suffices to show that $G \otimes a^{\otimes n}$ is a retract of $((G^\vee)^{\otimes n-1} \otimes G^{\otimes n}) \otimes a^{\otimes n}$ in $\text{DMF}(X, L, W)$.

We will show it by proving that G is a direct summand of $(G^\vee)^{\otimes n-1} \otimes G^{\otimes n}$ in $\mathrm{DMF}(X, L, W)$ by induction on n . The $n = 1$ case is trivial. For $n \geq 2$, assume that G is a direct summand of $(G^\vee)^{\otimes n-2} \otimes G^{\otimes n-1}$. It is enough to show that $(G^\vee)^{\otimes n-2} \otimes G^{\otimes n-1}$ is a direct summand of $(G^\vee)^{\otimes n-1} \otimes G^{\otimes n}$. But for $n \geq 3$ case, this follows from $n = 2$ case by tensoring $(G^\vee)^{\otimes n-3} \otimes G^{\otimes n-2}$. Therefore it suffices to prove that G is a direct summand of $G^\vee \otimes G^{\otimes 2}$. The tensor product $(-) \otimes G : \mathrm{DMF}(X, L, 0) \rightarrow \mathrm{DMF}(X, L, W)$ is left adjoint to $(-) \otimes G^\vee : \mathrm{DMF}(X, L, W) \rightarrow \mathrm{DMF}(X, L, 0)$. Let $\eta : \mathrm{id} \rightarrow (-) \otimes G \otimes G^\vee$ be its adjunction morphism. Then the functor morphism

$$(-) \otimes G \xrightarrow{\eta \otimes G} (-) \otimes G \otimes G^\vee \otimes G$$

is a split mono. Evaluating $\mathcal{O}_X \in \mathrm{DMF}(X, L, 0)$, we see that G is a direct summand of $G^\vee \otimes G^{\otimes 2}$. \square

5. CLASSIFICATION OF THICK SUBCATEGORIES OF $\mathrm{DMF}(X, L, W)$

In this section, we prove our main result. Let X be a separated Noetherian scheme, and let $W \in \Gamma(X, L)$ be any section of a line bundle L on X . At first, following [TT], we recall the definition of ample families of line bundles.

Definition 5.1. We say a quasi-compact and quasi-separated scheme S **has an ample family of line bundles** if there exists a family $\mathcal{L} := \{L_\alpha\}_{\alpha \in \mathcal{A}}$ of line bundles on S such that the family $\{S_f \mid f \in \Gamma(S, L_\alpha^{\otimes n}), L_\alpha \in \mathcal{L}, n > 0\}$ of open subsets form an open basis of S , where $S_f := \{p \in S \mid f(p) \neq 0\}$.

Remark 5.2. (1) Any scheme with an ample line bundle has an ample family of line bundles. In particular, any affine scheme has an ample family of line bundles. Any separated regular Noetherian scheme has an ample family of line bundles. See [TT, Example 2.1.2] for more examples of schemes with ample families of line bundles.

(2) If S has an ample family of line bundles, then S satisfies the resolution property by [TT, Lemma 2.1.3].

Proposition 5.3. *Let $Z \subseteq X$ be a closed subset of X . Assume that X has an ample family of line bundles. Then, the following holds.*

- (1) *There exists a matrix factorization $\mathcal{K} \in \mathrm{DMF}(X, L, 0)$ such that $\mathrm{Supp}(\mathcal{K}) = Z$.*
- (2) *If W is a non-zero-divisor and Z is contained in $\mathrm{Sing}(X_0/X)$, then there exists a matrix factorization $F \in \mathrm{DMF}(X, L, W)$ such that $\mathrm{Supp}(F) = Z$.*

Proof. (1) Since X has an ample family of line bundles, there are finitely many sections $f_i \in \Gamma(X, L_i)$ of line bundles L_i ($1 \leq i \leq l$) such that $Z = \bigcap_{i=1}^l Z(f_i)$ as closed subsets of X , where $Z(f_i)$ is the zero scheme of f_i . Let K_i be the object in $\mathrm{DMF}(X, L, 0)$ of the following form

$$K_i := \left(\mathcal{O}_X \xrightarrow{f_i} L_i \xrightarrow{0} L \right).$$

By Proposition 2.20.(1) and Lemma 2.23, we have $\mathrm{Supp}(K_i) = Z(f_i)$. If we set $\mathcal{K} := \bigotimes_{i=1}^l K_i$, then $\mathcal{K} \in \mathrm{DMF}(X, L, 0)$ and $\mathrm{Supp}(\mathcal{K}) = \bigcap \mathrm{Supp}(K_i) = Z$.

(2) Since X is Noetherian, we can decompose Z into finitely many irreducible components $Z = \bigcup_{i=1}^r Z_i$. Then, for each $1 \leq i \leq r$, there is a unique generic point $p_i \in Z_i$ of Z_i . By Proposition 3.5, there exists a matrix factorization $E_i \in \mathrm{DMF}(X, L, W)$ such that $p_i \in \mathrm{Supp}(E_i)$. Then $Z_i \subseteq \mathrm{Supp}(E_i)$, since p_i is a generic point of Z_i . By (1), there are matrix factorizations $\mathcal{K}_i \in \mathrm{DMF}(X, L, 0)$, for $1 \leq i \leq r$, such that $\mathrm{Supp}(\mathcal{K}_i) = Z_i$. Note that Lemma 2.24 implies the equality $\mathrm{Supp}(\mathcal{K}_i \otimes E_i) = \mathrm{Supp}(\mathcal{K}_i) \cap \mathrm{Supp}(E_i)$. If we set $F := \bigoplus_{i=1}^r (\mathcal{K}_i \otimes E_i)$, then $F \in \mathrm{DMF}(X, L, W)$ and we have

$$\mathrm{Supp}(F) = \bigcup_{i=1}^r \mathrm{Supp}(\mathcal{K}_i \otimes E_i) = \bigcup_{i=1}^r (\mathrm{Supp}(\mathcal{K}_i) \cap \mathrm{Supp}(E_i)) = Z.$$

This completes the proof. \square

Definition 5.4. Let $\mathcal{T} \subset \text{DMF}(X, L, W)$ be a triangulated full subcategory. We say that \mathcal{T} is \otimes -**submodule** if it is closed under tensor action of $\text{DMF}(X, L, 0)$, i.e. for any $F \in \text{DMF}(X, L, 0)$ and any $T \in \mathcal{T}$, we have $F \otimes T \in \mathcal{T}$. For an object $F \in \text{DMF}(X, L, W)$, we denote by

$$\langle F \rangle^\otimes \subset \text{DMF}(X, L, W)$$

the smallest thick \otimes -submodule containing F .

We prove the following proposition by using tensor nilpotence properties in the previous section.

Proposition 5.5. *For $E, F \in \text{DMF}(X, L, W)$, if $\text{Supp}(E) \subseteq \text{Supp}(F)$, then $E \in \langle F \rangle^\otimes$.*

Proof. Let $f : \mathcal{O}_X \rightarrow F^\vee \otimes F$ be the adjunction morphism in $\text{DMF}(X, L, 0)$ induced by the adjoint pair $(-) \otimes F \dashv \mathcal{H}om(F, -)$. Set $G := C(f)[-1]$, and let $a : G \rightarrow \mathcal{O}_X$ be a morphism which completes the following distinguished triangle

$$G \xrightarrow{a} \mathcal{O}_X \xrightarrow{f} F^\vee \otimes F \rightarrow G[1].$$

Then, since $\langle F \rangle^\otimes$ is closed under $\text{DMF}(X, L, 0)$ -action, $E \otimes C(a) \cong (E \otimes F^\vee) \otimes F \in \langle F \rangle^\otimes$. We claim that for any $n > 0$, $E \otimes C(a^{\otimes n}) \in \langle F \rangle^\otimes$. Indeed, consider the following diagram

$$\begin{array}{ccccccc} G^{\otimes n+1} & \xrightarrow{G \otimes (a^{\otimes n})} & G & \xrightarrow{G \otimes b_n} & G \otimes C(a^{\otimes n}) & \longrightarrow & G^{\otimes n+1}[1] \\ \parallel & \circlearrowleft & \downarrow a & & \downarrow & & \parallel \\ G^{\otimes n+1} & \xrightarrow{a^{\otimes n+1}} & \mathcal{O}_X & \xrightarrow{b_{n+1}} & C(a^{\otimes n+1}) & \longrightarrow & G^{\otimes n+1} \\ & & \downarrow & & \downarrow & & \\ & & C(a) & \xlongequal{\quad} & C(a) & & \\ & & \downarrow & & \downarrow & & \\ & & G[1] & \xrightarrow{G \otimes b_n[1]} & G \otimes C(a^{\otimes n})[1] & & \end{array}$$

where the top horizontal sequence is the distinguished triangle obtained by tensoring G with the following triangle

$$(*) \quad G^{\otimes n} \xrightarrow{a^{\otimes n}} \mathcal{O}_X \xrightarrow{b_n} C(a^{\otimes n}) \rightarrow G^{\otimes n}[1].$$

Then, by the octahedral axiom, we obtain the triangle completing the vertical sequence on the right side in the above diagram

$$G \otimes C(a^{\otimes n}) \rightarrow C(a^{\otimes n+1}) \rightarrow C(a) \rightarrow G \otimes C(a^{\otimes n})[1].$$

Considering the triangle obtained by tensoring this triangle with E , we can prove that $E \otimes C(a^{\otimes n}) \in \langle F \rangle^\otimes$ by induction on n .

Tensoring E with the above triangle $(*)$ for any $n > 0$, we have a triangle

$$E \otimes G^{\otimes n} \xrightarrow{E \otimes a^{\otimes n}} E \rightarrow E \otimes C(a^{\otimes n}) \rightarrow E \otimes G^{\otimes n}[1].$$

If $E \otimes a^{\otimes n} = 0$ for some $n > 0$, then E is a direct summand of $E \otimes C(a^{\otimes n}) \in \langle F \rangle^\otimes$, which implies that $E \in \langle F \rangle^\otimes$. Hence it suffices to show that there is an integer $n > 0$ such that $E \otimes a^{\otimes n} = 0$. By Lemma 4.3, it is enough to show that for any $p \in \text{Supp}(E)$, $a \otimes k(p) = 0$ in $\text{KMF}(k(p), 0)$. Since $p \in \text{Supp}(E) \subseteq \text{Supp}(F)$, $p \in \text{Supp}(F^\vee \otimes F)$. By Lemma 2.22, $F^\vee \otimes F \otimes k(p) \cong \mathcal{H}om_{k(p)}(F \otimes k(p), F \otimes k(p)) \neq 0$ in $\text{KMF}(k(p), 0)$. Hence the natural map

$$k(p) \xrightarrow{g} \mathcal{H}om_{k(p)}(F \otimes k(p), F \otimes k(p))$$

is a split mono by Corollary 2.17, since it is non-zero map. Since $f \otimes k(p) : k(p) \rightarrow F^\vee \otimes F \otimes k(p)$ is equal to the composition of $g : k(p) \rightarrow \mathcal{H}om_{k(p)}(F \otimes k(p), F \otimes k(p))$ and the natural isomorphism $\mathcal{H}om_{k(p)}(F \otimes k(p), F \otimes k(p)) \cong F^\vee \otimes F \otimes k(p)$, $f \otimes k(p)$ is also a split mono. Hence the triangle

$$G \otimes k(p) \xrightarrow{a \otimes k(p)} k(p) \xrightarrow{f \otimes k(p)} F^\vee \otimes F \otimes k(p) \rightarrow G \otimes k(p)[1]$$

implies that $a \otimes k(p) = 0$. □

Now we are ready to prove the following main result. Recall that a subset $S \subseteq T$ of a topological space T is called **specialization-closed** if it is a union of closed subsets of T . We easily see that S is specialization-closed if and only if $s \in S$ implies $\overline{\{s\}} \subseteq S$.

Theorem 5.6. *Let X be a separated Noetherian scheme with ample family of line bundles, L be a line bundle on X , and $W \in \Gamma(X, L)$ be a non-zero-divisor. There is one-to-one correspondence:*

$$\left\{ \begin{array}{l} \text{specialization-closed} \\ \text{subsets of } \text{Sing}(X_0/X) \end{array} \right\} \xrightarrow{\sigma} \left\{ \begin{array}{l} \text{thick } \otimes\text{-submodules} \\ \text{of } \text{DMF}(X, L, W) \end{array} \right\}$$

The bijective map σ sends Y to the thick subcategory consisting of matrix factorizations $F \in \text{DMF}(X, L, W)$ with $\text{Supp}(F) \subseteq Y$. The inverse bijection τ sends \mathcal{T} to the union $\bigcup_{F \in \mathcal{T}} \text{Supp}(F)$.

Proof. The map σ is well defined since for any $E \in \text{DMF}(X, L, 0)$ and $F \in \text{DMF}(X, L, W)$, we have $\text{Supp}(E \otimes F) \subseteq \text{Supp}(E) \cap \text{Supp}(F)$ by Lemma 2.24. The map τ is also well defined by Proposition 2.20.(2) and Proposition 3.5.(1).

We show that σ and τ are mutually inverse. Let Y be a specialization-closed subset of $\text{Sing}(X_0/X)$, and let \mathcal{T} be a thick \otimes -submodule of $\text{DMF}(X, L, W)$. By construction, we have $\tau(\sigma(Y)) \subseteq Y$ and $\mathcal{T} \subseteq \sigma(\tau(\mathcal{T}))$. Hence it is enough to show the inclusions $Y \subseteq \tau(\sigma(Y))$ and $\sigma(\tau(\mathcal{T})) \subseteq \mathcal{T}$.

Since $\text{Sing}(X_0/X)$ is a specialization-closed subset of X by Proposition 3.5.(1), Y is specialization-closed in X . Hence Y can be described as a union of closed subsets Y_λ of X ; $Y = \bigcup Y_\lambda$. By Proposition 5.3.(2), for each Y_λ , there exists $F_\lambda \in \text{DMF}(X, L, W)$ with $\text{Supp}(F_\lambda) = Y_\lambda$. Since $F_\lambda \in \sigma(Y)$, we have $Y_\lambda = \text{Supp}(F_\lambda) \subseteq \tau(\sigma(Y))$. Hence $Y \subseteq \tau(\sigma(Y))$.

To finish the proof, we show that $\sigma(\tau(\mathcal{T})) \subseteq \mathcal{T}$. Let $F \in \sigma(\tau(\mathcal{T}))$ be an object. Then, by construction, we have $\text{Supp}(F) \subseteq \bigcup_{T_\lambda \in \mathcal{T}} \text{Supp}(T_\lambda)$. Then, as in the proof of [Tho, Theorem 3.15], there is a finite set $\{T_\lambda\}_{\lambda \in \Lambda}$ of objects in \mathcal{T} such that $\text{Supp}(F) \subseteq \bigcup_{\lambda \in \Lambda} \text{Supp}(T_\lambda)$. Hence $\text{Supp}(F) \subseteq \text{Supp}(\bigoplus_{\lambda \in \Lambda} T_\lambda)$. Since $\bigoplus_{\lambda \in \Lambda} T_\lambda \in \mathcal{T}$, it follows from Proposition 5.5 that $F \in \mathcal{T}$. \square

Definition 5.7. We say that an object $G \in \text{DMF}(X, L, W)$ is a \otimes -**generator** of $\text{DMF}(X, L, W)$ if $\langle G \rangle^\otimes = \text{DMF}(X, L, W)$.

The following corollary says that the closedness of relative singular locus $\text{Sing}(X_0/X)$ in X_0 is related to the existence of a \otimes -generator of $\text{DMF}(X, L, W)$.

Corollary 5.8. *Notation is same as in Theorem 5.6. The relative singular locus $\text{Sing}(X_0/X)$ is closed in X_0 if and only if $\text{DMF}(X, L, W)$ has a \otimes -generator.*

Proof. Assume that the subset $\text{Sing}(X_0/X)$ is closed in X_0 . Since the relative singular locus $\text{Sing}(X_0/X)$ is the union of supports of all objects in $\text{DMF}(X, L, W)$ and X is Noetherian, there is a finite subset $\{F_i\}$ of objects $F_i \in \text{DMF}(X, L, W)$ such that $\text{Sing}(X_0/X) = \bigcup_i \text{Supp}(F_i) = \text{Supp}(\bigoplus_i F_i)$. By Theorem 5.6, there is a specialization-closed subset $Y \subseteq \text{Sing}(X_0/X)$ such that $\sigma(Y) = \langle \bigoplus_i F_i \rangle^\otimes$. Since $\text{Sing}(X_0/X) = \text{Supp}(\bigoplus_i F_i) \subseteq Y$, we have $Y = \text{Sing}(X_0/X)$. Hence $\langle \bigoplus_i F_i \rangle^\otimes = \sigma(\text{Sing}(X_0/X)) = \text{DMF}(X, L, W)$.

If $\text{DMF}(X, L, W)$ has a \otimes -generator G , for any object $F \in \text{DMF}(X, L, W)$, we have $\text{Supp}(F) \subseteq \text{Supp}(G)$ by Lemma 2.25. Hence Proposition 3.5.(1) implies that $\text{Sing}(X_0/X) = \text{Supp}(G)$, and so $\text{Sing}(X_0/X)$ is closed in X_0 . \square

Remark 5.9. Let (X, L, W) be the same LG model as in Theorem 5.6. If X is regular and L is ample, any thick subcategory of $\text{DMF}(X, L, W)$ is automatically \otimes -submodule. In particular, the set on the right-hand side in Theorem 5.6 is equal to the set of thick subcategories of $\text{DMF}(X, L, W)$. Since this fact is proved by Stevenson in a different context [Ste], we do not include the proof here.

6. TENSOR STRUCTURES ON MATRIX FACTORIZATIONS AND ITS SPECTRUM

Using the classification result in the previous section, we will construct the relative singular loci from derived matrix factorization categories by considering tensor structures induced by tensor products.

6.1. Balmer's tensor triangular geometry. Following [Bal] and [Yu1, Chapter 4], we will recall some basic definitions and results of the theory of tensor triangular geometry.

Definition 6.1. A **pseudo tensor triangulated category** (\mathcal{T}, \otimes) consists of a triangulated category \mathcal{T} and symmetric associated bifunctor $\otimes : \mathcal{T} \times \mathcal{T} \rightarrow \mathcal{T}$ which is exact in each variable. For the precise definition, see [Yu1, Definition 4.1.1].

Remark 6.2. We don't assume that a pseudo tensor triangulated category has a unit 1_\otimes , and this is the only difference from the original definition of tensor triangulated categories in [Bal].

Definition 6.3. Let (\mathcal{T}, \otimes) be a pseudo tensor triangulated category.

- (1) A thick subcategory $\mathcal{I} \subset \mathcal{T}$ is called **\otimes -ideal** if the following implication holds:

$$A \in \mathcal{T} \text{ and } B \in \mathcal{I} \Rightarrow A \otimes B \in \mathcal{I}.$$

- (2) A \otimes -ideal \mathcal{P} is called **prime** if the following holds

$$A \notin \mathcal{P} \text{ and } B \notin \mathcal{P} \Rightarrow A \otimes B \notin \mathcal{P}.$$

- (3) A \otimes -ideal \mathcal{I} is called **radical** if $\sqrt{\mathcal{I}} = \mathcal{I}$, where $\sqrt{\mathcal{I}}$ is the radical of \mathcal{I} , i.e.

$$\sqrt{\mathcal{I}} := \{ A \in \mathcal{T} \mid \exists n \geq 1 \text{ such that } A^{\otimes n} \in \mathcal{I} \}$$

For a pseudo tensor triangulated category (\mathcal{T}, \otimes) , we can consider the Zariski topology on the set of all prime ideals of (\mathcal{T}, \otimes) .

Definition 6.4. The **spectrum**, denoted by $\mathrm{Spc}(\mathcal{T}, \otimes)$, of (\mathcal{T}, \otimes) is defined as the set of all prime \otimes -ideals

$$\mathrm{Spc}(\mathcal{T}, \otimes) := \{ \mathcal{P} \mid \mathcal{P} \text{ is a prime } \otimes\text{-ideal} \}$$

The **Zariski topology** on $\mathrm{Spc}(\mathcal{T}, \otimes)$ is defined by the collection of closed subsets of the form $Z(\mathcal{S}) := \{ \mathcal{P} \in \mathrm{Spc}(\mathcal{T}, \otimes) \mid \mathcal{S} \cap \mathcal{P} = \emptyset \}$ for any family of objects $\mathcal{S} \subseteq \mathcal{T}$.

Definition 6.5. A **support data** on a pseudo tensor triangulated category (\mathcal{T}, \otimes) is a pair (X, σ) of a topological space X and an assignment $\sigma : \mathrm{Ob}\mathcal{T} \rightarrow \{\text{closed subsets of } X\}$ satisfying the following conditions:

- (1) $\sigma(0) = \emptyset$ and $\bigcup_{A \in \mathcal{T}} \sigma(A) = X$.
- (2) $\sigma(A \oplus B) = \sigma(A) \cup \sigma(B)$.
- (3) $\sigma(A[1]) = \sigma(A)$.
- (4) $\sigma(A) \subseteq \sigma(B) \cup \sigma(C)$ for any triangle $A \rightarrow B \rightarrow C \rightarrow A[1]$.
- (5) $\sigma(A \otimes B) = \sigma(A) \cap \sigma(B)$.

We say that a support data (X, σ) is **classifying** if the following properties hold:

- (a) The topological space X is Noetherian and any non-empty irreducible closed subset has a unique generic point.
- (b) There is the following bijective correspondence:

$$\Theta : \{ \text{specialization-closed subsets of } X \} \xrightarrow{\sim} \{ \text{radical thick } \otimes\text{-ideals} \}$$

$$\text{defined by } \Theta(Y) := \{ A \in \mathcal{T} \mid \sigma(A) \subseteq Y \}, \text{ with } \Theta^{-1}(\mathcal{I}) = \bigcup_{A \in \mathcal{I}} \sigma(A).$$

Remark 6.6. Because of the lack of the unit 1_\otimes in (\mathcal{T}, \otimes) in our setting, we replace the condition $\sigma(1_\otimes) = X$ in the original definition of support data in [Bal, Definition 3.1 (SD1)] with $\bigcup_{A \in \mathcal{T}} \sigma(A) = X$.

The following result is essentially due to Balmer [Bal], and it is the key result for the result in the next subsection. See also [Yu1, Theorem 4.1.16].

Theorem 6.7 ([Bal, Theorem 5.2]). *Assume that (X, σ) is a classifying support data on a pseudo tensor triangulated category (\mathcal{T}, \otimes) . Then we have the canonical homeomorphism*

$$f : X \xrightarrow{\sim} \mathrm{Spc}((\mathcal{T}, \otimes)),$$

defined by $f(x) := \{ A \in \mathcal{T} \mid x \notin \sigma(A) \}$.

6.2. Construction of relative singular loci from matrix factorizations. In this section, using our classification result, we construct relative singular loci from pseudo tensor triangulated structures on derived matrix factorization categories. This kind of observation is also discussed in [Yu2]. Following [Yu2], we consider the natural pseudo tensor triangulated structure on derived matrix factorization categories.

Throughout this section, X is a separated Noetherian scheme with ample family of line bundles, L is a line bundle on X , and $W \in \Gamma(X, L)$ is a non zero-divisor. Denote by X_0 the zero scheme of W , and assume that $2 \in \Gamma(X, \mathcal{O}_X)$ is a unit of the ring $\Gamma(X, \mathcal{O}_X)$.

For any unit $\lambda \in \Gamma(X, \mathcal{O}_X)^\times$ in the ring $\Gamma(X, \mathcal{O}_X)$, we have a natural functor

$$\lambda : \mathrm{DMF}(X, L, W) \rightarrow \mathrm{DMF}(X, L, \lambda W)$$

defined by $\lambda(F_1 \xrightarrow{\varphi_1} F_0 \xrightarrow{\varphi_0} F_1) := (F_1 \xrightarrow{\varphi_1} F_0 \xrightarrow{\lambda\varphi_0} F_1)$, and it is a triangulated equivalence (see [Yu1, Proposition 4.1.19]).

Definition 6.8. Suppose that $2 \in \Gamma(X, \mathcal{O}_X)$ is a unit. Then we define a bifunctor

$$\otimes^{\frac{1}{2}} : \mathrm{DMF}(X, L, W) \times \mathrm{DMF}(X, L, W) \rightarrow \mathrm{DMF}(X, L, W)$$

as the composition $\mathrm{DMF}(X, L, W) \times \mathrm{DMF}(X, L, W) \xrightarrow{\otimes} \mathrm{DMF}(X, L, 2W) \xrightarrow{\frac{1}{2}} \mathrm{DMF}(X, L, W)$.

As in [Yu1, Proposition 4.1.22], we see that the pair $(\mathrm{DMF}(X, L, W), \otimes^{\frac{1}{2}})$ is a pseudo tensor triangulated category.

Consider an assignment

$$\mathrm{Supp}(-) : \mathrm{Ob}(\mathrm{DMF}(X, L, W)) \rightarrow \{\text{specialization-closed subsets of } \mathrm{Sing}(X_0/X)\}$$

defined by the supports of objects in $\mathrm{DMF}(X, L, W)$.

Theorem 6.9. $(\mathrm{Supp}, \mathrm{Sing}(X_0/X))$ is a classifying support data on $(\mathrm{DMF}(X, L, W), \otimes^{\frac{1}{2}})$.

Proof. Since the functor $\frac{1}{2} : \mathrm{DMF}(X, L, 2W) \rightarrow \mathrm{DMF}(X, L, W)$ is an equivalence and commutes with taking stalks, we have

$$\mathrm{Supp}\left(\frac{1}{2}(F)\right) = \mathrm{Supp}(F).$$

Therefore, it follows from Lemma 2.25 and Proposition 3.5 that $(\mathrm{Supp}, \mathrm{Sing}(X_0/X))$ is a support data. We will show that the support data satisfies the conditions (a) and (b) in Definition 6.5.

We check the condition (a). Note that X satisfies the condition (a). It follows that $\mathrm{Sing}(X_0/X)$ is a Noetherian topological space, since so is X . Note that any *irreducible* closed subset Z of $\mathrm{Sing}(X_0/X)$ is closed in X . Indeed, since Z is closed in a specialization-closed subset of X , Z is specialization-closed in X . Hence Z is a union of irreducible closed subsets Z_λ of X ; $Z = \bigcup_\lambda Z_\lambda$. Since Z is irreducible in $\mathrm{Sing}(X_0/X)$, there is an irreducible closed subset $Z_{\lambda'}$ of X such that $Z = Z_{\lambda'}$. Hence Z is an irreducible closed subset of X , and it has a unique generic point.

Next, we verify the condition (b). By Theorem 5.6, it is enough to show that for a thick subcategory \mathcal{T} of $\mathrm{DMF}(X, L, W)$

$$\mathcal{T} \text{ is } \otimes\text{-submodule} \Leftrightarrow \mathcal{T} \text{ is radical } \otimes^{\frac{1}{2}}\text{-ideal}.$$

The implication (\Rightarrow) follows immediately from Theorem 5.6 and Lemma 2.25.(4). We show the other implication (\Leftarrow) . For this, let $\mathcal{I} \subset \mathrm{DMF}(X, L, W)$ be a radical thick $\otimes^{\frac{1}{2}}$ -ideal. For objects $E \in \mathrm{DMF}(X, L, 0)$ and $F \in \mathcal{I}$, it suffices to prove that $E \otimes F \in \mathcal{I}$. We have

$$\begin{aligned} (E \otimes F) \otimes^{\frac{1}{2}} (E \otimes F) &= \frac{1}{2}((E \otimes F) \otimes (E \otimes F)) \\ &\cong \frac{1}{2}((E \otimes F \otimes E) \otimes F) \\ &= (E \otimes F \otimes E) \otimes^{\frac{1}{2}} F, \end{aligned}$$

and the object in the bottom line is in \mathcal{I} since $E \otimes F \otimes E \in \mathrm{DMF}(X, L, W)$ and \mathcal{I} is $\otimes^{\frac{1}{2}}$ -ideal. Hence $E \otimes F \in \mathcal{I}$ as \mathcal{I} is radical. \square

Theorem 6.7 and Theorem 6.9 imply the following result.

Corollary 6.10. *There is a homeomorphism*

$$\mathrm{Spc}(\mathrm{DMF}(X, L, W), \otimes^{\frac{1}{2}}) \cong \mathrm{Sing}(X_0/X)$$

Remark 6.11. By Proposition 2.12 and Proposition 3.2, we see that Corollary 6.10 is a generalization of [Yu2, Theorem 1.2], where Yu consider the case when X is a regular affine scheme of finite Krull dimension.

REFERENCES

- [BFK] M. Ballard, D. Favero and L. Katzarkov, *A category of kernels for equivariant factorizations and its implications for Hodge theory*, Publ. Math. Inst. Hautes Études Sci. **120** (2014), 1-111.
- [BDFIK] M. Ballard, D. Deliu, D. Favero, M. U. Isik, and L. Katzarkov, *Resolutions in factorization categories*, Adv. in Math. **295** (2016), 195-249.
- [Bal] P. Balmer, *The spectrum of prime ideals in tensor triangulated categories*, J. Reine Angew. Math. **588** (2005), 149-168.
- [EP] A. I. Efimov and L. Positselski, *Coherent analogues of matrix factorizations and relative singularity categories*, Algebra Number Theory **9** (2015), no. 5, 1159-1292.
- [Gab] P. Gabriel, *Des catégories abéliennes*, Bull. Soc. Math. France, **90** (1962), 323-448.
- [LS] V. A. Lunts and O. M. Schnürer, *Matrix factorizations and semi-orthogonal decompositions for blowing-ups*, J. Noncommut. Geom. **10** (2016), no. 3, 907-979.
- [Mat1] *Commutative algebra*, Second edition, Mathematics Lecture Note Series, 56. Benjamin/Cummings Publishing Co., Inc., Reading, Mass., (1980).
- [Mat2] H. Matsumura, *Commutative ring theory*, second ed. Cambridge Studies in Advanced Mathematics, vol. 8. Cambridge University Press, Cambridge (1989).
- [Orl1] D. O. Orlov, *Triangulated categories of singularities and D-branes in Landau-Ginzburg models*, Tr. Mat. Inst. Steklova, **246** (2004), 240-262. In Russian; translated in Proc. Steklov Math. Inst. **246**, (2004), 227-248.
- [Orl2] D. Orlov, *Matrix factorizations for nonaffine LG-models*, Math. Ann. **353** (2012), no. 1, 95-108.
- [Pos] L. Positselski, *Two kinds of derived categories, Koszul duality, and comodule-contramodule correspondence*, Mem. Amer. Math. Soc. **212** (2011), no. 966.
- [Ros] A. L. Rosenberg, *The spectrum of abelian categories and reconstruction of schemes*, Rings, Hopf algebras, and Brauer groups (Antwerp/Brussels, 1996), 257-274, Lecture Notes in Pure and Appl. Math., 197, Dekker, New York, 1998.
- [Ser] J.-P. Serre, *Géométrie algébrique et géométrie analytique*, Ann. Inst. Fourier, Grenoble, **6** (1956), 1-42.
- [Ste] G. Stevenson, *Subcategories of singularity categories via tensor actions*, Compos. Math. **150** (2014), no. 2, 229-272.
- [Tak] R. Takahashi, *Classifying thick subcategories of the stable category of Cohen-Macaulay modules*, Adv. Math. **225** (2010), no. 4, 2076-2116.
- [Tho] R. W. Thomason, *The classification of triangulated subcategories*, Compos. Math. **105** (1997), 1-27.
- [TT] R. W. Thomason and T. Trobaugh, *Higher algebraic K-theory of schemes and of derived categories*, The Grothendieck Festschrift, Vol. III, 247-435, Progr. Math., vol. 88, Birkhäuser Boston, Boston, MA 1990.
- [Yu1] Xuan Yu, *Geometric study of the category of matrix factorizations*, ProQuest LLC, Ann Arbor, MI, 2013. Thesis (Ph.D.)-The University of Nebraska-Lincoln. MR3187316.
- [Yu2] Xuan Yu, *The triangular spectrum of matrix factorizations is the singular locus*, Proc. Amer. Math. Soc., **144** (2016), no. 8, 3283-3290.

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